

## The Approximation of Fourth Order Nonlinear Variational Inequalities

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**ABSTRACT**

In this paper we deal with existence and the approximation of the solution of the elliptic variational inequality from an abstract axiomatic point of view. We give an error estimate for the difference of the two solutions in an appropriate norm. Also, we present some computational results.

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**Received:** September 02, 2023; **Accepted:** September 06, 2023; **Published:** September 20, 2023**Introduction**

An important and very useful class of nonlinear problems arising from mechanics, physics, engineering, etc. consists of the so-called variational inequalities. These are of two types, the elliptic variational inequalities and the parabolic variational inequalities (see, Duvant et al. and Lions [1,2]. In this paper we shall study the uniqueness, existence, and approximation of the solution of elliptic variational inequality, and also determine an error estimate for the difference of the two solutions in an appropriate norm. Several authors have discussed the variational inequalities of elliptic type. In Brezis and Stampacchia have considered the elliptic variational inequality of the form:

Find  $\phi \in K$  such that

$$(f - A\phi, \psi - \phi) \leq 0, \quad \forall \psi \in K, \quad (1)$$

where  $A$  is a monotone, semicontinuous operator from  $V$  to  $V'$  and  $K$  is a closed convex subset of  $V$ , a real Hilbert space, and  $f \in V'$  [3]. They have proved an abstract regularity theorem for this general problem and then apply it to some convex sets. In [4], Lions and Stampacchia have studied that if  $K$  is a closed convex set of  $V$  and  $a(\phi, \psi)$  is coercive, that is

$$a(\phi, \phi) \geq \alpha \|\phi\|^2, \quad \forall \phi \in V,$$

and some constant  $\alpha > 0$ , then the problem (1) has a unique solution. Some work on the approximation of the solutions of the elliptic variational inequalities has previously been done by Mosco and Aubin [5,6]. In Mosco has considered the approximation of the solutions of problem (1), taking into account perturbations of " $a$ ", the bilinear form,  $f$ , the element of  $V'$ , and  $K$ , the closed convex set [5]. In Aubin also considered the approximation of the solutions of problem (1) [6].

The problem we consider lies in the following general setting. Let  $\Omega$  be a bounded open domain in  $\mathbb{R}^n$  with boundary  $\Gamma$ . Let  $V$  be the real Hilbert space with scalar product  $(\cdot, \cdot)$  and associated norm  $\|\cdot\|$ . Let  $V'$  be the dual of  $V$ , and the pairing between  $V$  and  $V'$  denoted by  $(\cdot, \cdot)$ . We consider an operator  $A : V = H_0^2(\Omega) \rightarrow V'$ , which is linear,

continuous and symmetric, and satisfies the coercivity condition, that is

$$(A\phi, \phi) = a(\phi, \phi) \geq \alpha \|\phi\|^2, \quad \forall \phi \in V, \quad (2)$$

for instance, we may take,  $A = -\Delta$ . Let now consider the Sobolev space  $H^2(\Omega)$  (which is the set of square integrable functions with square integrable first derivatives) of real-valued function such that

$$\psi \in L^2(\Omega), \quad \partial\psi/\partial x_i \in L^2(\Omega), \quad i = 1, 2, \dots, n,$$

where  $L^2(\Omega)$  is the space of (classes of) square summable function on  $\Omega$ , and the derivative  $\partial\psi/\partial x_i$  are taken in the sense of distributions on  $\Omega$ . Define

$$H_0^2(\Omega) = \{\psi | \psi \in H^2(\Omega), \psi = 0 \text{ on } \Gamma\},$$

which is a closed vector subspace of  $H^2(\Omega)$  and  $f$  an element of  $V'$ . Finally, let  $K$  be a convex subset of  $V$  and is defined as follows

$$K = \{\psi | \psi \in H_0^2(\Omega), \psi \geq 0 \text{ on } \Omega\}. \quad (3)$$

We want to approximate the solution to problem of the following:

**The Continuous Problem**

Consider the elliptic variational inequality of the following form

$$\phi \in K \quad (A\phi, \psi - \phi) + \mu(\phi, \psi - \phi) \geq (f, \psi - \phi), \quad \forall \psi \in K, \quad (4)$$

where  $\mu (> 0)$  is constant. The problem in which we are interested is finding the solution of (4) in the closed convex set  $K$ . In Butt has proved the existence and uniqueness of the solution of the penalized differential equation of the form

$$A\phi_\epsilon + \mu\phi_\epsilon + 1/\epsilon\phi_\epsilon^- = f, \quad \mu > 0,$$

(where  $\phi_\epsilon^- = -\sup(-\phi_\epsilon, 0)$ ), and showed that, its solution

$\phi_\epsilon$  tends to the solution of the variational inequality (4) when tends to zero [7]. Now we shall prove the existence and uniqueness of the solution of (4) in the form of the following theorem:

**Theorem A:** Let  $A$  be a linear continuous and symmetric operator satisfying (2), and let  $K$  be a closed convex set of  $V$ . There exists a unique solution of (4). The map:  $f \rightarrow \phi$  is continuous from  $V'$  into  $V$ .

**Proof:** To prove the uniqueness of the problem, let  $\phi$  and  $\psi$  be two possible solutions of (4); if we take  $\psi = \phi$  in (4) (respectively,  $\phi = \psi$  in the equation analogous to (4) with respect to  $\psi$ ), we have

$$\phi \in K \quad (A\phi, \psi - \phi) + \mu(\phi, \psi - \phi) \geq (f, \psi - \phi), \quad \forall \psi \in K, \quad (5)$$

and

$$\phi \in K \quad (A\psi, \phi - \psi) + \mu(\psi, \phi - \psi) \geq (f, \phi - \psi), \quad \forall \psi \in K, \quad (6)$$

now, by adding (5) and (6), we obtain:

$$(A(\phi - \psi), \psi - \phi) + \mu(\phi - \psi, \psi - \phi) \geq 0,$$

or

$$(A(\psi - \phi), \psi - \phi) + \mu(\psi - \phi, \psi - \phi) \leq 0,$$

by using (2), we have

$$\alpha\|\psi - \phi\|^2 + \mu\|\psi - \phi\|^2 \leq (A(\psi - \phi), \psi - \phi) + \mu(\psi - \phi, \psi - \phi) \leq 0,$$

which implies that

$$C\|\psi - \phi\|^2 \leq 0,$$

where  $C$  is constant, and since  $C > 0$ , therefore, we obtain  $\psi = \phi$ . Let us denote  $A \in L(V; V')$  the map dened, and by setting  $\|A\| = M$  and  $\|\mu\| = N$ . Let  $\Upsilon$  be the canonical isomorphism from  $V'$  onto  $V$  defined, for  $f \in V'$ , by

$$(f, \varphi) = ((\Upsilon f, \varphi)) \quad \text{for } \varphi \in V, \quad (7)$$

then

$$\|\Upsilon\|_{L(V, V')} = \|\Upsilon^{-1}\|_{L(V, V')} = 1.$$

Before proving the existence part of the problem, we prove a lemma.

**Lemma A:** Let  $\lambda$  be a number such that  $0 < \lambda < \frac{2\gamma}{Q}$ .

Then there exists  $\theta$  with  $0 < \theta < 1$  such that

$$|((\phi, \varphi)) - \lambda(A\phi, \varphi) + \mu(\phi, \varphi)| \leq \theta\|\phi\|\|\varphi\|, \quad \forall \phi, \varphi \in V. \quad (8)$$

**Proof:** Infact

$$|((\phi, \varphi)) - \lambda(A\phi, \varphi) + \mu(\phi, \varphi)| = |((\phi - \lambda\Upsilon(A\phi + \mu\phi), \varphi))| \leq \|\phi - \lambda\Upsilon(A\phi + \mu\phi)\|\|\varphi\|.$$

On the other hand,

$$\begin{aligned} \|\phi - \lambda\Upsilon(A\phi + \mu\phi)\|^2 &= \|\phi\|^2 + \lambda^2\|\Upsilon(A\phi + \mu\phi)\|^2 - 2\lambda\Upsilon(A\phi + \mu\phi)\phi \\ &= \|\phi\|^2 + \lambda^2[\|\Upsilon A\phi\|^2 + \|\Upsilon\mu\phi\|^2 \\ &\quad + 2\mu(A\phi, \varphi)] - 2\lambda\Upsilon(A\phi + \mu\phi)\phi \\ &= \|\phi\|^2 + \lambda^2\|A\|^2\|\phi\|^2 + \lambda^2\|\mu\|^2\|\phi\|^2 \\ &\quad + 2\lambda^2\mu\alpha\|\phi\|^2 - 2\lambda(\alpha\|\phi\|^2 + \mu\|\phi\|^2) \\ &= (1 + \lambda^2(M^2 + N^2) + 2\lambda^2\mu\alpha - 2\lambda\alpha - 2\lambda\mu)\|\phi\|^2 \\ &= (1 + \lambda^2(M^2 + N^2 + 2\mu\alpha) - 2\lambda(\alpha + \mu))\|\phi\|^2 \\ &= (1 + \lambda^2Q - 2\lambda\gamma)\|\phi\|^2, \end{aligned}$$

where  $Q = (M^2 + N^2 + 2\mu\alpha)$ , and  $\gamma = (\alpha + \mu)$ . Since  $\gamma > 0$ , we can choose  $\lambda$  such that

$$(1 + \lambda^2Q - 2\lambda\gamma)^{1/2} < 1,$$

and hence, (8) follows.

Now we shall prove the existence part of the problem.

(i) Suppose that,  $(A\phi, \varphi) = ((\phi, \varphi))$ . Then the problem (4) means:

Find  $\phi \in K$  such that

$$\begin{aligned} ((\phi, \varphi - \phi)) + \mu((\Upsilon\phi, \varphi - \phi)) &\geq (f, \varphi - \phi) \\ &= ((\Upsilon f, \varphi - \phi)), \quad \forall \varphi \in K, \end{aligned} \quad (9)$$

that is

$$((\phi + \Upsilon\mu\phi - \Upsilon f, \varphi - \phi)) \geq 0.$$

If  $\phi$  satisfies (9), then

$$\|\phi - \Upsilon(f - \mu\phi)\| \leq \|\varphi - \Upsilon(f - \mu\phi)\|,$$

and conversely. Since  $K$  is a closed convex set, therefore, there exists one and only one element of  $K$  which minimizes

$$\|\varphi - \Upsilon(f - \mu\phi)\|, \quad \forall \varphi \in K.$$

The solution of variational inequality (4) is thus given by

$$\phi = P_K\Upsilon(f - \mu\phi),$$

where  $P_K$  is the projection of  $V$  in  $K$ .

(ii) Consider the general case: Fix  $\lambda$  as in the above lemma  $A$ .

For  $\phi \in V$ , define  $\Phi(\phi) \in V'$  and setting

$$(\Phi(\phi), \varphi) = ((\phi, \varphi)) - \lambda[(A\phi, \varphi) + \mu(\phi, \varphi)] + \lambda(f, \varphi), \quad \forall \varphi \in V.$$

Then, for  $\phi_1, \phi_2 \in V$ , and by using the above lemma  $A$ , we have

$$\begin{aligned} |(\Phi(\phi_1) - \Phi(\phi_2), \varphi)| &= |((\phi_1 - \phi_2, \varphi)) - \lambda| \\ &\quad [(A(\phi_1 - \phi_2), \varphi) + \mu(\phi_1 - \phi_2, \varphi)]| \end{aligned} \quad (10)$$

Then by (i), there exists a unique  $\omega \in K$  such that

$$((\omega, \varphi - \omega)) + \mu((\omega, \varphi - \omega)) \geq (\Phi(\phi), \varphi - \omega), \quad \varphi \in K,$$

and  $\omega$  is given by

$$\omega = P_K\Upsilon\Phi(\phi) = T\phi, \quad (11)$$

the problem reduces to finding the fixed points of  $T$ . This defines a map  $\phi \rightarrow T\phi$  from  $V$  into  $K$ . Moreover,

$$\|T\phi_1 - T\phi_2\| = \|P_K \Upsilon \Phi(\phi_1) - P_K \Upsilon \Phi(\phi_2)\| \leq \|\Upsilon \Phi(\phi_1) - \Upsilon \Phi(\phi_2)\| \leq \|\Phi(\phi_1) - \Phi(\phi_2)\| \leq \theta \|\phi_1 - \phi_2\|.$$

Since  $\theta < 1$ ,  $T\phi$  is a contraction and thus there exists one and only one solution  $\phi$  such that

$$T\phi = \phi. \quad (12)$$

Then  $\phi$  belongs to  $K$  and satisfies

$$((\phi, \varphi - \phi)) \geq (\Phi(\phi), \varphi - \phi) = ((\phi, \varphi - \phi)) - \lambda[(A\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi)] - (f, \varphi - \phi), \quad \forall \varphi \in K. \quad (13)$$

It follows ( $\lambda > 0$ ) that  $\phi$  is a solution of (4). For more details, see Glowinski et al. and Lions et al. [8,4].

### The Approximation Problem

Let us suppose we are given a parameter  $h$  converging to zero and a family  $\{V_h\}_{0 < h \leq 1}$  of closed subspaces of  $V$  (in practice, the  $V_h$  are finite dimensional and the parameter  $h$  varies over a sequence). We are also given a family  $\{K_h\}_h$  of closed convex nonempty subset of  $V$ . An approximation of (4) generally involves seeking a function  $\phi_h$  in a set  $K_h$  which is a subset of a finite dimensional subspace  $V_h$  of  $V$ ,  $h$  being an appropriate index. The approximation of (4) will then involve seeking  $\phi_h \in K_h$  such that

$$(A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h), \quad \forall \varphi_h \in K_h. \quad (14)$$

We are interested in determining sequences of solutions  $\{\phi_h\}$  to (14),  $\phi_h \in K_h \subset V_h$ , and in investigating the behavior of the approximations as  $h \rightarrow 0$ . In particular, we want to determine conditions under which  $\{\phi_h\}$  converges in some sense to a solution to (4) and estimating the error  $\phi - \phi_h$ . For a consistent approximation of the set  $K_h$  we use the following conditions: Let  $K$  be a subset of a space  $V$ . A sequence of subsets  $\{K_h\}_h$  in  $V_h$  is said to converge to a set  $K$  if:

a) for every weakly convergent sequence  $\{\phi_h\}_h$ ,  $\phi_h \in K_h$ , for all  $h$ , its weak limit  $\phi$  belongs to  $K$ .

b) There exists  $\zeta \subset V$ ,  $\bar{\zeta} = K$  and  $\lambda : \zeta \rightarrow K_h$  such that

$$\lim_{h \rightarrow 0} \lambda_h \phi = \phi, \text{ strongly in } V, \forall \phi \in \zeta.$$

Now we shall prove the theorem for seeking the solution of (14).

**Theorem B:** Prove that the problem (14) has a unique solution.

**Proof:** We can prove this theorem in the similar way as we proved the theorem A. In theorem A, taking  $V$  to be  $V_h$  and  $K$  to be  $K_h$ , we have the result. Now we prove the convergence results in the following way:

**Theorem C:** Let  $K$  is a closed convex nonempty subset of  $V$ , and  $\{K_h\}_h$  be a family of closed convex nonempty subset of  $V$ . Then, we have

$$\lim_{h \rightarrow 0} \|\phi_h - \phi\|_V = 0, \quad (15)$$

where  $\phi_h$  is the solution of (14) and  $\phi$  the solution of (4).

**Proof:** To prove this kind of convergence results, first, we shall obtain a priori estimates for  $\{\phi_h\}_h$ , then the weak convergence of  $\{\phi_h\}_h$  and finally, with the help of the weak convergence, we shall prove the strong convergence. For obtaining the estimates for  $\phi_h$  we shall show that there exists two constants, called,  $\beta_1$  and  $\beta_2$  independent of  $h$ , such that

$$\|\phi_h\|^2 \leq \beta_1 \|\phi_h\| + \beta_2, \quad \forall h. \quad (16)$$

Since we know that  $\phi_h$  is the solution of (14), we have

$$(A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h),$$

or,

$$(A\phi_h, \phi_h) + \mu(\phi_h, \phi_h) \leq (A\phi_h, \varphi_h) + \mu(\phi_h, \varphi_h) - (f, \varphi_h - \phi_h),$$

by using (2), we have

$$\alpha \|\phi_h\|^2 + \|\mu\| \|\phi_h\|^2 \leq \|A\| \|\phi_h\| \|\varphi_h\| + \|\mu\| \|\phi_h\| \|\varphi_h\| + \|f\| (\|\phi_h\| + \|\varphi_h\|), \quad \forall \varphi_h \in K_h. \quad (17)$$

Let  $\varphi_0 \in \zeta$  and  $\varphi_h = \lambda_h \varphi_0 \in K_h$ , then by the condition on  $K_h$

(that is, (b)), we have,  $\lambda_h \varphi_0 \rightarrow \varphi_0$  strongly in  $V$  and hence  $\|\varphi_h\|$

is uniformly bounded by a constant  $m$ . Hence (17) can be written as

$$\|\phi_h\|^2 \leq 1/C \{ (m\|A\| + m\|\mu\| + \|f\|) \|\phi_h\| + m\|f\| \}, \quad (18)$$

$$\|\phi_h\|^2 \leq \beta_1 \|\phi_h\| + \beta_2,$$

where  $\beta_1 = 1/C \{ m\|A\| + m\|\mu\| + \|f\| \}$ ,  $\beta_2 = m/C \|f\|$ ,

and  $C (> 0)$  is constant. Then, (16) implies that

$$\|\phi_h\| \leq C, \quad \forall h,$$

where here, and what follows,  $C$  denotes the various constant.

Now we shall discuss the weak convergence of  $\{\phi_h\}_h$ . Since the relation (16) implies that  $\phi_h$  is uniformly bounded. Hence there exists a subsequence, say,  $\{\phi_{h_i}\}$  such that  $\phi_{h_i}$  converges to  $\Phi^*$  weakly in  $V$ . Then, by the condition of  $\{K_h\}_h$  (that is, (a)), we have  $\Phi^* \in K$ . Now we shall prove that  $\Phi^*$  is a solution of (4). Since we know that

$$(A\phi_{h_i}, \phi_{h_i}) \leq (A\phi_{h_i}, \varphi_{h_i}) + \mu(\phi_{h_i}, \varphi_{h_i} - \phi_{h_i}) - (f, \varphi_{h_i} - \phi_{h_i}), \quad (19)$$

by taking  $\varphi_{h_i} = \lambda_{h_i} \varphi$ , and  $\varphi \in \zeta$ , then (19) becomes:

$$(A\phi_{h_i}, \phi_{h_i}) \leq (A\phi_{h_i}, \lambda_{h_i} \varphi) + \mu(\phi_{h_i}, \lambda_{h_i} \varphi - \phi_{h_i}) - (f, \lambda_{h_i} \varphi - \phi_{h_i}). \quad (20)$$

Since  $\lambda_{h_i} \varphi$  converges strongly to  $\varphi$  and  $\phi_{h_i}$  converges to  $\Phi^*$  weakly as  $h_i$  tends to zero, so by taking the limit as  $h_i$  tends to zero in (19), we have

$$\liminf_{h_i \rightarrow 0} (A\phi_{h_i}, \phi_{h_i}) \leq (A\Phi^*, \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*), \quad \forall \varphi \in \zeta. \quad (21)$$

Since  $\phi_{h_i} \rightarrow \Phi^*$  weakly, so by using the known theorem [9] and from the continuity of "a", and the fact that

$$\liminf_{h_i \rightarrow 0} (A\phi_{h_i}, \phi_{h_i}) \geq (A\Phi^*, \Phi^*), \quad (22)$$

so, from (21) and (22), we obtain

$$(A\Phi^*, \Phi^*) \leq \liminf_{h_i \rightarrow 0} (A\phi_{h_i}, \phi_{h_i}) \leq (A\Phi^*, \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*), \quad \forall \varphi \in \zeta.$$

Therefore, we have

$$(A\Phi^*, \Phi^* - \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*) \geq 0, \quad \forall \varphi \in \zeta, \Phi^* \in K. \quad (23)$$

Since  $\zeta$  is dense in  $K$  and  $A$  is continuous, so from (23), we get

$$(A\Phi^*, \Phi^* - \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*) \geq 0, \quad \forall \varphi \in K, \Phi^* \in K. \quad (24)$$

Hence  $\Phi^*$  is a solution of (4). Since, by theorem A, the solution of (4) is unique and so  $\Phi^* = \phi$  is the unique solution. Hence  $\phi$  is the only limit point of  $\{\phi_h\}_h$  in the weak topology of  $V$ . Hence the whole  $\{\phi_h\}_h$  converges to  $\phi$  weakly.

Now we shall discuss the strong convergence. By using (2), we have

$$0 \leq \alpha \|\phi_h - \phi\|^2 \leq (A\phi_h - \phi, \phi_h - \phi) = (A\phi_h, \phi_h) - (A\phi_h, \phi) - (A\phi, \phi_h) + (A\phi, \phi), \quad (25)$$

where  $\phi_h$  is solution of (14) and  $\phi$  is solution of (4). Since  $\phi_h$  is solution of (14) and  $\lambda_h \varphi \in K_h$  for any  $\varphi \in \zeta$ , so we get (14) of the form

$$(A\phi_h, \phi_h) \leq (A\phi_h, \lambda_h \varphi) + \mu(\phi_h, \lambda_h \varphi - \phi_h) - (f, \lambda_h \varphi - \phi_h), \quad \forall \varphi \in \zeta. \quad (26)$$

Since we know that

$$\lim_{h \rightarrow 0} \phi_h = \phi, \text{ weakly, in } V,$$

and

$$\lim_{h \rightarrow 0} \lambda_h \varphi = \varphi, \text{ strongly; in } V \text{ (by condition (b));}$$

we obtain (26) from (25), and after taking the limit, we have

$$0 \leq \alpha \liminf \|\phi_h - \phi\|^2 \leq \alpha \limsup \|\phi_h - \phi\|^2 \leq (A\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi) - (f, \varphi - \phi), \quad \forall \varphi \in \zeta. \quad (27)$$

By density and continuity, (27) also holds for all  $\varphi \in K$ ; then taking  $\varphi = \phi$  in (27), we get

$$\lim_{h \rightarrow 0} \|\phi_h - \phi\| = 0, \quad (28)$$

which proves the strong convergence.

We shall now determine the error estimate of the difference of the solutions of related variational inequalities for the appropriate norm in the form of the following theorem.

**Theorem D:** Let us consider an operator  $A : V = H_0^2(\Omega) \rightarrow V'$ , which is linear, continuous and symmetric, and satisfies the coercivity condition (2), and let  $\phi \in K$  and  $\phi_h \in K_h$  be respective

solutions of the variational inequalities:

$$\phi \in K \quad (A\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi) \geq (f, \varphi - \phi), \quad \forall \varphi \in K, \quad (29)$$

and

$$K_h \quad (A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h), \quad \forall \varphi_h \in K_h, \quad (30)$$

where  $K$  and  $K_h$  are closed convex subsets of  $H_0^2(\Omega)$ .

Let  $A$  be the operator such that

$$A\phi = - \sum_{i,j} \partial^2 / \partial x_i^2 (a_{ij}(x) \partial^2 / \partial x_j^2 \phi),$$

where

$$a_{ij}(x) \in L^\infty(\Omega), \text{ and } \sum_{i,j} a_{ij}(x) \omega_i \omega_j \geq \alpha \sum_{i,h} \omega_i^2, \alpha > 0, \quad \forall \omega_i \in \mathfrak{R}^n.$$

If  $(f - (A\phi + \mu\phi)) \in L^2(\Omega)$ , then

$$\|\phi - \phi_h\| \leq \{2/\alpha \|f - A\phi - \mu\phi\| \inf_{\varphi \in K} \|\phi_h - \varphi\| + \inf_{\varphi_h \in K_h} [2/\alpha \|f - A\phi - \mu\phi\| \|\phi - \varphi_h\| + C^2/\alpha^2 \|\phi - \varphi_h\|^2]\}^{1/2}.$$

**Proof:** Adding the inequalities (29) and (30), we obtain

$$(A\phi, \phi - \varphi) + (A\phi_h, \phi_h - \varphi_h) + \mu(\phi, \phi - \varphi) + \mu(\phi_h, \phi_h - \varphi_h) - (f, \phi - \varphi) + (f, \phi_h - \varphi_h),$$

or

$$(A\phi, \phi) - (A\phi, \phi_h) - (A\phi_h, \phi) + (A\phi_h, \phi_h) + \mu(\phi, \phi) - \mu(\phi, \phi_h) - \mu(\phi_h, \phi) + \mu(\phi_h, \phi_h) \leq (A\phi, \varphi) + (A\phi_h, \varphi_h) - (A\phi_h, \phi) + (A\phi, \phi_h) + \mu(\phi, \varphi) + \mu(\phi_h, \varphi_h) - \mu(\phi_h, \phi) - \mu(\phi, \phi_h) + (f, \phi - \varphi) + (f, \phi_h - \varphi_h)$$

After, regrouping, we obtain

$$(A(\phi - \phi_h), \phi - \phi_h) + \mu(\phi - \phi_h, \phi - \phi_h) \leq (f, \phi - \varphi_h) + (f, \phi_h - \varphi) + (A\phi, \varphi - \phi_h) + (A\phi_h, \varphi_h - \phi) + \mu(\phi, \varphi - \phi_h) + \mu(\phi_h, \varphi_h - \phi) = (f, \phi - \varphi_h) + (f, \phi_h - \varphi) + (A\phi, \varphi - \phi_h) + (A\phi_h, \varphi_h - \phi) + (A(\phi - \phi_h), \phi - \phi_h) + \mu(\phi - \phi_h, \phi - \phi_h) + \mu(\phi, \varphi - \phi_h) + \mu(\phi_h, \varphi_h - \phi).$$

Since

$$\mu(\phi - \phi_h, \phi - \phi_h) \geq 0,$$

so, by using the hypotheses of the theorem, we have

$$\alpha \|\phi - \phi_h\|^2 \leq \|f - A\phi - \mu\phi\| \|\phi - \varphi_h\| + \|f - A\phi - \mu\phi\| \|\phi_h - \varphi\| + C \|\phi_h - \phi\| \|\phi - \varphi_h\| + \|\mu\| \|\phi_h - \phi\| \|\phi - \varphi_h\|,$$

or

$$\alpha \|\phi - \phi_h\|^2 \leq \|f - A\phi - \mu\phi\| \|\phi - \varphi_h\| + \|f - A\phi - \mu\phi\| \|\phi_h - \varphi\| + C \|\phi_h - \phi\| \|\phi - \varphi_h\|,$$

where  $C$  denotes various constant. Since we know (Young's inequality) that

$$ab \leq a^2/2\epsilon + \epsilon b^2/2, \quad \text{for all } \epsilon > 0,$$

so we make the following estimate:

$$C\|\phi - \varphi_h\| \|\phi_h - \phi\| \leq C^2/2\alpha\|\phi - \varphi_h\|^2 + \alpha/2\|\phi - \phi_h\|^2,$$

so we have, for all  $\varphi \in K$  and  $\varphi_h \in K_h$

$$\|\phi - \phi_h\| \leq \{2/\alpha(\|f - A\phi - \mu\phi\| \|\phi - \varphi_h\| + \|f - A\phi - \mu\phi\| \|\phi_h - \varphi\|) + C^2/\alpha^2\|\phi - \varphi_h\|^2\}^{1/2},$$

which implies that

$$\|\phi - \phi_h\| \leq \{2/\alpha\|f - A\phi - \mu\phi\| \inf_{\varphi \in K} \|\phi_h - \varphi\| +$$

$$\inf_{\varphi_h \in K_h} [2/\alpha\|f - A\phi - \mu\phi\| \|\phi - \varphi_h\| + C^2/\alpha^2\|\phi - \varphi_h\|^2]\}^{1/2}. \quad (31)$$

Now we shall discuss about the computation work by taking a simple example of one-dimensional.

### Computational Results

In order to solve the approximate variational inequality (4), we use the fact that the problem (4) is equivalent to the problem for finding  $\phi \in K$  which minimizes

$$1/2(A\varphi, \varphi) - (\chi, \varphi), \quad \forall \varphi \in K, \quad (32)$$

where  $\chi = f - \mu\phi$ . It is very easy to prove this equivalence, since

$$I(\varphi) = 1/2(A\varphi, \varphi) - (\chi, \varphi), \quad \forall \varphi \in K,$$

is convex function of  $\varphi$  on  $K$  and every local minimum of  $I(\varphi)$  in  $K$  is also a global minimum of  $I(\varphi)$  in  $K$ . So we can have existence and uniqueness for the minimization problem. For more details see [8]. Now we shall consider a one-dimensional version of the problem of seepage flow through a homogeneous rectangular dam. We shall solve this problem numerically using the fixed point method [8]. The example problem involves finding a solution  $\phi \in K$  of a one-dimensional variational inequality

$$\int_0^1 \phi'(\varphi - \phi)' dx + \mu \int_0^1 (\varphi - \phi) dx \geq \int_0^1 f(\varphi - \phi) dx, \quad \forall \varphi \in K, \quad (33)$$

where  $\phi' = d\phi/dx$ , and let

$$K = \{\varphi \in H^2(0, 1) : \varphi(0) = 1/4, \varphi(1) = 0 \text{ \& } \varphi(x) \geq 0 \text{ in } (0, 1)\}. \quad (34)$$

Let the domain  $\Omega = (0; 1)$  be discretized by a uniform mesh containing  $N$  finite elements. Within a finite element, every function  $\varphi \in H^2(\Omega)$  is approximated by function of the form

$$\varphi = \varphi^1 \Psi_1(\theta) + \varphi^2 \Psi_2(\theta),$$

where  $\varphi^i$  is the value of the  $i$ th local nodal point,  $\Psi_i$  is the local interpolation function at  $i$ th nodal point, and  $\theta$  is a local coordinate in the finite element. For a unit linear element

$$\Psi_1(\theta) = 1 - \theta, \quad \text{and} \quad \Psi_2(\theta) = \theta.$$

Upon assembling the elements, a global modal is obtained for which every function  $\varphi \in H^2(\Omega)$  is approximated by

$$\varphi_h = \sum_{i=1}^n \varphi_i \psi^i(x), \quad (35)$$

where  $\varphi_i$  is the value of  $\varphi_h$  at  $i$ th (global) nodal point, and  $\psi^i(x)$  is the global basis function corresponding to the  $i$ th (global) nodal point. Introducing the approximation (35) into (33) and (34), the variational inequality (33) on the admissible set (34) reduces to an optimization problem in  $\mathfrak{R}^n$

$$\phi_j \psi_j' \cdot (\varphi_i \psi_i' - \phi_i \psi_i') + \mu \phi_i \psi_j (\varphi_i \psi_i - \phi_i \psi_i) - f(\varphi_i \psi_i - \phi_i \psi_i) \geq 0,$$

or, for  $\{\phi_i\} \in \mathfrak{R}_h$

$$(\varphi_i - \phi_i)(K^{ij} \phi_j + G^{ij} \phi_j - F^i) \geq 0, \quad \forall \{\varphi_i\} \in \mathfrak{R}_h,$$

where repeated indices are summed and

$$\mathfrak{R}_h = \{\{\varphi_i\} \in \mathfrak{R}^n : \varphi_1 = 1/4, \varphi_N = 0 \text{ \& } \varphi_i \geq 0,$$

$$i = 2, \dots, N - 1\}, \quad (37)$$

and

$$K^{ij} = \int_0^1 \psi_i' \cdot \psi_j' dx, \quad G^{ij} = \mu \int_0^1 \psi_i \psi_j dx, \\ \text{and } F^i = -f \int_0^1 \psi_i dx, \quad (38)$$

and  $N$  is the total number of nodal points in the finite element model. We shall solve (36) and (37) using the fixed point method. The iterative scheme we used, is denoted in general as

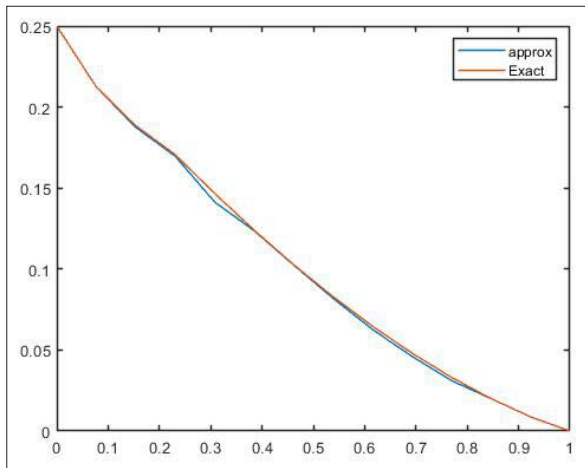
$$\phi_i^{t+1} = \max(0, \phi_i^t - \lambda_i (\sum_{j=1}^{i-1} (K^{ij} + G^{ij}) \phi_j^{t+1} + \sum_{j=i}^n (K^{ij} + G^{ij}) \phi_j^{t+1} - F^i)), \quad (39)$$

where the projection  $P_k$  is denoted pointwise by

$$P_k(\{\cdot\}) = \{\max(0, \cdot)\}, \quad \text{with } 0 < \lambda < \frac{2\gamma}{Q}.$$

The terminal value  $\phi_j^{t+1}$ ,  $1 \leq j \leq i$  are used in calculating the value  $\phi_i^{t+1}$ . We solve the variational inequality (36) for  $N = 20$ ,

with  $\mu = f = 1$ , by using fixed point method. The exact solution and approximate finite element results on the variational inequality (36) are shown in the figure below. Also, in the below table, we showed the numerical results obtained by using the fixed point scheme and compared with the exact solution of (36). According to this, we found that the numerical results are same to the exact solution up to five significant figures.



**Table: Comparison of the Solutions with Iteration**

Node:	Aproxi. Soln.	Exact Soln.	Error
1	0.2500000	0.2500000	0.0000000
2	0.2123777	0.2123787	0.0000010
3	0.1884904	0.1884915	0.0000011
4	0.1706367	0.1706378	0.0000011
5	0.1466371	0.1466384	0.0000013
6	0.1239702	0.1239721	0.0000019
7	0.1027054	0.1027072	0.0000018
8	0.0829138	0.0829156	0.0000018
9	0.0646738	0.0646728	0.0000010
10	0.0480571	0.0480583	0.0000012
11	0.0331544	0.0331555	0.0000011
12	0.0200515	0.0200521	0.0000006
13	0.0088398	0.0088405	0.0000007
14	0.0000000	0.0000000	0.0000000

factor = 0:05, at iteration 14.

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