Ricci Solitons on $\alpha$-para Kenmotsu Manifolds with Semi Symmetric Metric Connection

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ABSTRACT

In this paper we introduce notion of Ricci solitons in $\alpha$-para Kenmotsu manifold with semi-symmetric metric connection. We have found the relations between curvature tensor, Ricci tensors and scalar curvature of $\alpha$-para Kenmotsu manifold with semi-symmetric metric connection. We have proved that 3-dimensional $\alpha$-para Kenmotsu manifold with semi-symmetric metric connection is an $\eta$-Einstein manifold and the Ricci soliton defined on this manifold is named expanding and steady with respect to the value of $\lambda$ constant. It is proved that Conharmonically flat $\alpha$-para Kenmotsu manifold with semi-symmetric metric connection is $\eta$-Einstein manifold.

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Introduction

In 1972 Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions and this manifolds is known as Kenmotsu manifold [1]. Sharma and Sinha started to study of the Ricci solitons in contact geometry in 1983 [2]. Later Mukut Mani Tripathi, Cornelia Livia Bejan and Mircea Crasmareanu, and others extensively studied Ricci solitons in contact metric manifolds [3, 4]. In 1985, almost paracontact geometry was introduced by Kaneyuki and Williams and then it was continued by many authors [5]. Nagaraja ve premalatha studied exclusively about Ricci solitons on Kenmotsu manifold in 2012. Agashe and Chafle, Liang, Pravonovic and Sengupta, Yildiz and Cetinkaya studied semi-symmetric non-metric connection in different ways [6-11].

A systematic study of almost paracontact metric manifolds was carried out by Zamkovoy [12]. However such structures were also studied by Buchner and Rosca, Rosca and Venbecke [13]. Further almost Para-Hermitian Structure on the tangent of an almost Para-Co hermitian manifolds was studied by Bejan [3]. A class of $\alpha$-para kenmotsu manifolds was studied by Srivastva and Srivastva [14]. We can observe that the concircular curvature tensor on Pseudo-Riemannian manifold to be of constant curvature. Hayden introduced Semi-symmetric linear connection on a Riemannian manifold [8]. Let M be an n-dimensional Riemannian manifold of class C-endowed with the Riemannian metric g and $\overline{\nabla}$be the Levi-Civita Connection on $M^n$. A linear connection $\overline{\nabla}$ defined on $M^n$ is said to be semi-symmetric if its torsion tensor T is of the form [15].

where $\xi$ is a vector field and $\eta$ is a 1-form defined by

$$g(X, \xi) = \eta(X)$$

for all vector field X $\in \chi(M^n)$ where, $\chi(M^n)$ is the set of all differentiable vector fields on $M^n$. A relation between the semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection $\nabla$ on $M^n$ has been obtained by Yano which is given as [16]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi$$

(1.1)

Preliminaries

A differentiable manifold $M^n$ of dimension n is said to have an almost paracontact $(\phi, \xi, \eta)$-structure if it admits an (1,1) tensor field $\phi$, a unique vector field $\xi$, and 1-form $\eta$ such that :

$$\phi^2 = I - \eta \otimes \xi, \quad \phi \eta = 0, \quad \eta \circ \phi = 0$$

(2.1)

$$\eta(\xi) = 1$$

(2.2)

for any vector field X, Y on $M^n$. The manifold $M^n$ equipped with an almost paracontact structure $(\phi, \xi, \eta)$ is called almost paracontact manifold. In addition, if an almost paracontact manifold admits a pseudo-Riemannian metric satisfying...
\[g(X, \xi) = \eta(X)\]  
\[(2.3)\]

\[g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)\]  
\[(2.4)\]

\[g(\phi X, Y) = -g(X, \phi Y)\]  
\[(2.5)\]

for any vector field \(X, Y\) on \(M^n\), where \(\phi\) is a \((1,1)\) tensor field, \(\xi\) is a vector field, \(\eta\) is a 1-form and \(g\) is the Riemannian metric. Then \(M^\nu\) is called an almost contact manifold. For an almost contact manifold \(M^\nu\), it follows that [8]

\[(\Box_X \phi) Y = \Box_X \phi Y - \phi(\Box_X Y)\]  
\[(2.6)\]

\[(\Box_X \eta) Y = -\Box_X \eta(Y) - \eta(\Box_X Y)\]  
\[(2.7)\]

Let \(R\) be Ricci curvature tensor, \(S\) Ricci curvature tensor, \(Q\) Ricci operator we have

\[S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X), e_i)\]  
\[(2.8)\]

\[QX = -\sum_{i=1}^{n} R(e_i, X)e_i\]  
\[(2.9)\]

and

\[S(X, Y) = g(QX, Y)\]  
\[(2.10)\]

for any vector field \(X, Y\) on \(M^n\). Then the manifold is called paracontact metric structure and the manifold \(M^n\) equipped with an almost paracontact metric structure is called an almost paracontact metric manifold. Further in addition, if the structure \((\phi, \xi, \eta, g)\) satisfies

\[d\eta(X, Y) = g(X, \phi Y)\]  
\[(2.11)\]

for any vector fields \(X, Y\) on \(M^n\). Then the manifold is called paracontact metric manifold and the corresponding structure \((\phi, \xi, \eta, g)\), is called a paracontact structure with the associated metric \(g\) [17]. On an almost paracontact metric manifold, the \((1,2)\) tensor field \(N_\phi\) defined as

\[N_\phi = [\phi, \phi] - 2d\eta \otimes \xi\]  
\[(2.12)\]

Where \([\phi, \phi]\) is the nijenhuis tensor of \(\phi\). If \(N\) vanishes identically, then we say that the manifold \(M^n\) is a normal almost paracontact metric manifold. The normality condition implies that the almost paracomplex structure \(J\) defined on \(M^n\times\mathbb{R}\)

\[J(X, t\frac{d}{dt}) = (\phi X + \lambda(\xi), \eta(X)\frac{d}{dt})\]

is integrable. Here \(X\) is tangent to \(M^n\), \(t\) is the coordinate of \(\mathbb{R}\) and \(\lambda\) is a differentiable function on \(M^n\times\mathbb{R}\).

For an almost paracontact metric 3-dimensional manifold \((\phi, \xi, \eta, g)\), the following are conditions are mutually equivalent:

(i) there exist smooth functions \(\alpha, \beta\) on \(M^\nu\) such that

\[\Box_X \xi = \alpha(X - \eta(X)\xi) + \beta\phi X\]  
\[(2.14)\]

where \(\Box\) is the Levi-Civita connection of a pseudo-Riemannian metric \(g\). A normal almost paracontact metric 3-dimensional manifold is called

(a) Para-Cosymplectic manifold if \(\alpha = \beta = 0\),

(b) quasi para-Sasakian manifold if and only if \(\alpha = 0\) and \(\beta \neq 0\),

(c) \(\beta\)-para Sasakian manifold if and only if \(\alpha = 0\) and \(\beta\) is a non-zero constant, in particular para-Sasakian manifold if \(\beta = -1\)

(d) \(\alpha\)-para Kenmotsu manifold if \(\alpha\) is a non-zero constant and \(\beta = 0\) in particular para-Kenmotsu manifold if \(\alpha = 1\).

On 3-Dimensional \(\alpha\)-para Kenmotsu Manifold with Semi-Symmetric Metric Connection

In 3-dimensional \(\alpha\)-para Kenmotsu manifold, the Ricci tensor \(S\) of Levi-Civita connection \(g\) is given by

\[S(X, Y) = g(R(e_1, Y), e_1) - g(R(\phi e_1, Y), e_1) + g(R(\xi, X), Y)\]

Let \(M^\nu\) be an \(\alpha\)-para Kenmotsu manifold [13], then we have

\[R(X, e_1) = \left(\xi + 2\alpha^2\eta\right)g(Y, X) - g(X, Y)\]

\[QX = -2\alpha^2\eta\]  
\[(2.1)\]

Replace \(Z = \xi\) in equation (3.1), we get

\[R(X, e_1)\xi = \alpha^2\eta(X)Y - \eta(Y)X,\]
\[(3.2)\]

\[S(X, Y) = \left(\xi + 2\alpha^2\eta\right)g(X, Y) - \left(\xi + 3\alpha^2\eta\right)\eta(Y)\]
\[(3.3)\]

\[S(X, \xi) = -2\alpha^2\eta(X)\]
\[(3.4)\]

\[\Box_X \phi Y = \alpha\phi g(X, Y)\xi - \eta(Y)\phi X\]
\[(3.5)\]

\[\Box_X \xi = \alpha(X - \eta(X)\xi)\]
\[(3.6)\]

Let \(\Box\) be a linear connection and \(\Box\) be a Riemann connection of an \(\alpha\)-para Kenmotsu manifold \(M\). This \(\Box\) linear connection defined by

\[\Box_X Y = \Box_X Y + \eta(Y)X - g(X, Y)\xi\]  
\[(3.7)\]

For \(\alpha\)-para Kenmotsu manifold with semi-symmetric metric connection, using (2.6), (3.5), and (3.7) we have

\[[\Box_X \phi] Y = \alpha\left[g(\phi X, Y)\xi - \eta(Y)\phi X\right] + \eta(Y)\phi X\]
\[(3.8)\]

from equation (3.7), we have

\[\Box_X \xi = (1 + \alpha)(X - \eta(X)\xi)\]
\[(3.9)\]

Let \(M^\nu\) be a 3-dimensional \(\alpha\)-para Kenmotsu manifold. The curvature tensor \(R\) of \(M^\nu\) with respect to the semi-symmetric metric connection \(\Box\) is defined by

\[R(X, Y)Z = \Box_X \Box_Y Z - \Box_Y \Box_X Z - \Box_{[X,Y]} Z\]  
\[(3.10)\]

with the help of (3.7) and (3.9), we get
By using equations (3.7), (2.2), (2.3), (3.6), (3.9) (3.10), (3.11), (3.12) and (3.13), we get

\[ g(\bar{R}(X,Y)Z, U) = g(\bar{R}(X,Y)Z, U) \]

In (3.15) taking the inner product with Z, we have

\[ g(\bar{R}(X,Y)Z, Z) = \alpha (1 + \alpha) g(Y, Z) \]

Thus we have

\[ g(\bar{R}(X,Y)Z, Z) = \alpha (1 + \alpha) (\eta(X) g(Y, Z) - 2\eta(Y) g(X, Z)) \]

Lemma 3.1 Let M be a 3-dimensional α-para Kenmotsu manifold with the semi-symmetric metric connection, \( \bar{S} \) Ricci curvature tensor and \( \bar{Q} \) Ricci operator. Then

\[ \bar{S}(X, \xi) = -2\alpha(1 + \alpha)\eta(X) \]

and

\[ \bar{Q}\xi = -2\alpha(1 + \alpha)\xi \]

Proof. Contracting with Y and Z in (3.17) and summing over i=1,2,...,n, from (2.8) expression

\[ \sum g(\bar{R}(e_i, Y)\xi, e_i) = \alpha (1 + \alpha) (\sum \eta(e_i) g(Y, e_i) - \eta(Y) \sum g(e_i, e_i)) \]

the proof of (3.18) is completed. Then also using (2.10) and (2.1), (2.2), (2.3), the proof of (3.19) is completed.

Lemma 3.2 Let M be a 3-dimensional α-para Kenmotsu manifold with the semi-symmetric metric connection, \( \bar{S} \) scalar curvature tensor, \( \bar{S}(X, Y) \) Ricci curvature tensor and \( \bar{Q} X \) Ricci operator. Then it follows that

\[ \bar{S}(X, \xi) = (1 + \alpha) \bar{S}(X, Y) + 2\alpha \eta(X) \eta(Y) \]

and

\[ \bar{Q}\xi = (1 + \alpha) \bar{Q} X \]

Proof. Taking inner product of equation (3.14) with Y and using equation (2.3) we have

\[ g(\bar{R}(X,Y)Z, U) = g(\bar{R}(X,Y)Z, U) \]

Replace \( Y = \xi \) in equation (3.15) and using equation (2.3), we get

\[ \bar{Q}\xi = (1 + \alpha) \bar{Q} X \]

Let \( \{e_i, \phi e_i, \xi\} \) be a local orthonormal \( \phi \)-basis of vector fields on α-para Kenmotsu manifold \( M \). Then, we get

\[ g(\bar{R}(X,Y)Z, U) = g(\bar{R}(X,Y)Z, U) \]

where \( \bar{S} \) is the scalar curvature with semi-symmetric metric connection. Using (3.23) and (2.10), it’s verified that

\[ g(\bar{R}(X,Y)Z, U) = g((1 + \alpha) (\eta(X) g(Y, Z) - 2\eta(Y) g(X, Z)) \]

the proof of (3.21) is completed.

Ricci Solitons in α-para Kenmotsu Manifold with Semi-Symmetric Metric Connection

Let M be a 3-dimensional α-para Kenmotsu manifold with the semi-symmetric metric connection and V be pointwise collinear with \( \xi \) (i.e.V = b\xi, where b is a function). Then

\[ (LV g + 2\bar{S} + 2\bar{Q} g)(X, Y) = 0 \]

implies

\[ 0 = b\eta(\bar{S}X) + (Xb)\eta(Y) + b\eta(\bar{Q}X) \]

(4.1)

using (3.9) in (4.1), we get

\[ 0 = 2b(1 + \alpha) g(Y, Z) - 2b(1 + \alpha) \eta(X) \eta(Y) + (Xb)\eta(Y) \]

(4.2)

With the substitution of Y with \( \xi \) in (4.2), it follows that

\[ (Xb) + (Xb)\eta(Y) + 2\eta(\bar{Q}X) - 4\alpha(1 + \alpha) b(X) = 0 \]

(4.3)

Again replacing X by \( \xi \) in (4.3) shows that

\[ \xi b - \lambda + 2\alpha(\alpha + 1) \]

(4.4)
Putting (4.4) in (4.3), we obtain
\[ b = (2\alpha(1 + \alpha) - \lambda)\eta \]  
(4.5)

By applying \( d \) in (4.5), we get
\[ 0 = (2\alpha(1 + \alpha) - \lambda)d\eta \]  
(4.6)

Since \( d\eta \neq 0 \) from, we have
\[ 2\alpha(1 + \alpha) - \lambda = 0 \]  
(4.7)

By using (4.5) and (4.7), we obtain that \( b \) is constant. Hence from (4.2) it is verified
\[ S(X,Y) = -b((1 + \alpha) + \lambda)g(X,Y) + b(1 + \alpha)\eta(X)\eta(Y) \]  
(4.8)

which implies that \( M \) is an \( \eta \)-Einstein manifold. This leads to the following

**Theorem 4.1** If in a 3-dimensional \( \alpha \)-para Kenmotsu manifold with the semi symmetric metric connection, the metric \( g \) is a Ricci soliton and \( V \) is a pointwise collinear with \( \xi \), then \( V \) is a constant multiple of \( \xi \) and \( g \) is an \( \eta \)-Einstein manifold of the form (4.8) and Ricci soliton is steady and expanding according as \( \lambda = 2\alpha(1 + \alpha) \) is positive, respectively.

**Conharmonically Flat \( \alpha \)-para Kenmotsu Manifolds with the Semi-Symmetric Metric Connection**

We have studied conharmonically flat \( \alpha \)-para Kenmotsu manifolds with respect to the semi-symmetric metric connection. In a \( \alpha \)-para Kenmotsu manifold the conharmonic curvature tensor with respect to the semi-symmetric metric connection is given by
\[ \bar{R}(X,Y)Z = \bar{R}(X,Y)Z - [S(Y,Z)X - S(X,Z)Y] + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y. \]  
(5.1)

If \( \bar{R} = 0 \), then the manifold \( M \) is called conharmonically flat manifold with respect to the semi-symmetric metric connection. Let \( M \) be a conharmonically flat manifold with respect to the semi-symmetric metric connection. From (5.1), we have
\[ R(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y \]  
(5.2)

Now putting \( X = \xi \) in (5.3), we obtain
\[ \bar{R}(\xi,Y)Z = S(Y,Z)\xi - S(\xi,Z)Y + (\frac{C}{2} - 1 - 4\alpha)g(Y,Z)\xi - \eta(Z)\eta(Y)\xi, \]  
(5.4)

using (3.1) and (3.4) in (5.4), we get
\[ S(Y,Z)\xi - S(\xi,Z)Y + (-1 - 4\alpha - 2\alpha^2)g(Y,Z)\xi + (1 + 4\alpha + 2\alpha^2)\eta(Z)\eta(Y) = 0 \]  
(5.5)

Taking inner product with \( \xi \) in (5.5), we get
\[ S(Y,Z) = (1 + 4\alpha + 2\alpha^2)g(Y,Z) - (1 + 4\alpha + 4\alpha^2)\eta(Y)\eta(Z) \]  
(5.6)

Thus \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection. This leads to the following

**Theorem 5.1** If \( M \) is a conharmonically flat \( \alpha \)-para Kenmotsu manifolds with respect to the semi-symmetric metric connection. Then the manifold \( M \) is an \( \eta \)-Einstein.

**Example**

(A 3-dimensional \( \alpha \)-para Kenmotsu manifold with the semi-symmetric metric connection.) We consider the 3-dimensional manifold \( M = (x,y,z) \in R^3, z \neq 0 \), where \( (x,y,z) \) are the standard coordinates in \( R^3 \). The vector fields
\[ e_1 = z^2 \frac{\partial}{\partial x}, e_2 = z^2 \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z} \]
are linearly independent at each point of \( M \). Let \( g \) be the Riemannian metric defined by
\[ g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \]
\[ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the \((1,1)\) tensor field
\[ \phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0. \]

Then using linearity of \( \phi \) and \( g \) we have
\[ \eta(e_3) = 1, \phi(Z) = -Z + \eta(Z)e_3 \]
\[ g(\phi(Z), W) = g(Z, W) - \eta(Z)\eta(W) \]
for any \( Z, W \in \chi(M) \). Now, by direct computations we obtain
\[ [e_1, e_2] = 0, [e_2, e_3] = -\frac{z}{2}e_2, [e_1, e_3] = -\frac{z}{2}e_1 \]

by using these above equations we get [1]
\[ \bar{e}_1, e_3 = \frac{z}{2}e_3 \text{ and } \bar{e}_2, e_3 = -\frac{z}{2}e_1 \]
\[ \bar{e}_3, e_1 = \bar{e}_3, e_2 = \bar{e}_3, e_3 = 0 \]

Now we consider at this example for semi-symmetric metric connection. From (3.8), (6.1) and (6.2)
\[ \bar{e}_1, e_3 = \frac{z}{2} - 1)e_3 \text{ and } \bar{e}_3, e_3 = (-\frac{z}{2} + 1)e_1 \]
\[ \bar{e}_3, e_3 = 0 \text{ and } \bar{e}_3, e_3 = 0 \]

where \( i \neq j = 1,2 \). it’s known that
\[ \bar{R}(e_1, e_3)e_3 = \frac{6}{2} + \frac{2}{2}e_1, \bar{R}(e_1, e_3)e_3 = 0 \]
\[ \bar{R}(e_1, e_1)e_1 = \frac{4}{z} - \frac{4}{z} - 1)e_1, \bar{R}(e_1, e_3)e_3 = 0 \]
\[ \bar{R}(e_3, e_1)e_1 = \frac{3}{z} - \frac{6}{z^2}e_3 \]  
(6.6)
where $i \neq j = 1,2$. From (2.8) and (6.6) it's verified that

\[
S(e_1, e_1) = \left( \frac{-2}{x^2} + \frac{2}{z} - 1 \right)
\]

\[
S(e_2, e_2) = \left( \frac{-10}{x^2} + \frac{6}{z} - 1 \right)
\]

(6.7)

\[
S(e_3, e_2) = \left( \frac{-12}{x^2} + \frac{4}{z} \right)
\]

**Conclusion**

If in a 3-dimensional $\alpha$-para Kenmotsu manifold with the semi-symmetric metric connection, the metric $g$ is a Ricci soliton and in this study, we gave some curvature conditions for 3-dimensional $\alpha$-para Kenmotsu manifolds with semi-symmetric metric connection. In 3-dimensional $\alpha$-para Kenmotsu manifolds with semi-symmetric metric connection is also an $\eta$-Einstein manifold and Ricci soliton defined steady or expanding on this manifold is named with respect to values of $\alpha$ and $\lambda$ constant. We also proved that conharmonically flat $\alpha$-para Kenmotsu manifolds with semi-symmetric metric connection is an $\eta$-Einstein manifold.

**References**