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Quantum Entanglement and its Quantification in a Two-Mode Cascade Laser

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ABSTRACT

In this paper, quantum entanglement of correlated two-mode light generated by a three-level laser coupled to a two-mode squeezed vacuum reservoir is thoroughly analyzed using different inseparability criteria, using the master equation, we obtain the stochastic dierential equation and the correlation properties of the noise forces associated with the normal ordering. Next, we study the photon entanglement by considering different inseparability criteria. In particular, the criteria applied are Duan-Giedke-Cirac-Zoller (DGCZ) criterion, logarithmic negativity, Hillery-Zubairy, and Cauchy-Schwartz inequality and we found that the presence of the squeezing parameter leads to an increase in the degree of entanglement. Moreover, the linear gain coecient significantly achieved the degree of entanglement for the cavity radiation.

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Introduction

Quantum entanglement has been considered as the nonlocality aspect of quantum correlations with no classical similarity This wonderful feature was investigated in the seminal paper of Einstein-Podolsky-Rosen (EPR) [1]. After that, Bell recognized that entanglement leads to experimentally testable deviations of quantum mechanics from classical physics [2]. Furthermore, with the advent of quantum information theory, entanglement was known as a resource for many applications such as quantum cryptography quantum computation and communication quantum dense coding quantum teleportation entanglement swapping sensitive measurements and quantum telecloning [3-9]. Hence, an interest of understanding entanglement creation and quantification has gained the attention of several authors [10-12]. A three-level cascade laser has a great deal of interest over the years in connection with its potential as a source of the strong correlated photons exhibiting various non-classical properties. One of the possible mechanisms of producing this strong correlation is linked to atomic coherence that can be induced by preparing the atoms initially in a coherent super- position of the up and down levels [13, 14]. In this regard, three-level lasers can be defined as a twophoton quantum optical device that produces a strong correlated light with some non-classical features such as squeezing and entanglement which are the subject of this paper. This study shows that a quantum optical system can generates light in squeezed state under certain conditions. Tesfa S. has studied the entanglement amplification and squeezing properties of the cavity mode produced by non degenerate three level laser applying the solution of the stochastic dierential equation when the atomic coherence is introduced initially preparing three level atom by a coherent superposition of the top and bottom level via the intermediate as his result relatively a better squeezing is found when sufficiently large number of atoms are injected in to the cavity and a significant entanglement between the state of light generated in the cavity of the non-degenerate three level cascade laser, due to the strong correlation between the radiation emitted when the atom decays from the top level to the bottom level via the intermediate level [15].

Furthermore, studied the entanglement quantification of correlated photons generated by three level laser with parametric amplifier and coupled to two mode vacuum reservoir, and they showed that the degree of entanglement studied by logarithmic negativity and DGCZ criteria is greatly enhanced by increasing the rate of atomic injection when the atomic coherence is closer to its maximum value. In both cases, the weak entangled light is generated when all atoms are initially prepared in the lower energy state and a large number of atoms are constantly injected into the cavity regardless of the amplitude of the parametric amplifier [16]. On the other hand, the important eect of the parametric amplifier occurs in both of these approaches for smaller values of the linear coecient. Finally they conclude that, the two-mode light is found to be entangled even for the minimum and maximum atomic coherence which, respectively, corresponds to the absence and availability of more photons in the cavity.

In this paper, the entanglement of the light produced by a non-degenerate three-level laser coupled to a twomode squeezed vacuum reservoir is studied. Different criteria of entanglement quantification are used to detect the entanglement of the two-mode light generated by the two-photon optical device under consideration. We carry out our analysis applying the pertinent master equation describing the dynamics of the optical device. Using the resulting solutions, the degree of entanglement for the cavity radiation using the Duan-Giedke-Cirac-Zoller, logarithmic negativity, Hillery-Zubairy and Cauchy-Schwartz inequality inseparability criteria are measured [17-20].

The Model and Hamiltonian

We represent the top, intermediate, and bottom levels of a three-level atom in a cascade configuration by $|a>$, $|b>$, and $|c>$, respectively, as shown in figure below.

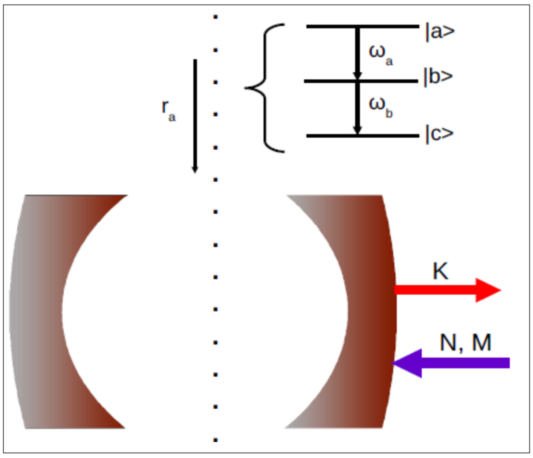

Figure 1: Schematic representation of a non-degenerate three level laser coupled to two squeezed vacuum reservoir, where r_a is the rate of atomic injection into the cavity, κ (kappa) is cavity damping constant and the constants N and M , which describe to the effect of the external environment, are related to each other through the squeeze parameter r as $M = \sqrt{N(N+1)}$ in which $N = sinhar.$ Hence, the squeezing parameter quantifies the mean photon number of the two-mode squeezed vacuum reservoir and inter-modal correlations among the reservoir sub-modes

In addition, we assume the two modes a and b to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |a\rangle$, respectively and direct transition between level $|a\rangle$ and level $|c\rangle$ to be dipole forbidden and also we consider the case in which three level atoms in the cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity at a constant rate r_a (rate of atomic injection in to the cavity) and removed after some time τ which is long enough for the atoms to decay spontaneously to levels other than the middle level or the lower level. The linear approximation which preserves the quantum properties of the radiation is applied and then the atomic variables are adiabatically eliminated keeping the eective cavity-mode equation containing the remaining atomic variables in the good cavity limit. When an atom makes a transition between the top and bottom level via the intermediate level two correlated photon of non-degenerate frequencies,are generated as shown on the above figure. We assume that these transition frequencies are at resonance with the two nondegenerate cavity modes.

The quantum optical system outlined in the above figure can be described in the interaction picture by the Hamiltonian

$$\hat{H} \;=\; \hat{H}_C + \hat{H}_R, \tag{1}$$

where

$$\hat{H}_C = ig[\hat{a}^{\dagger}|b\rangle\langle a| - |a\rangle\langle b|\hat{a} + \hat{b}^{\dagger}|c\rangle\langle b| - |b\rangle\langle c|\hat{b}]. \tag{2}$$

is the Hamiltonian describing the interactions of the three level atom with cavity mode, g is the atom-cavity mode coupling constant assumed to be the same for both transition, $\hat{a}$ and $\hat{b}$ are annihilation operators for the two cavity modes.

Similarly, the Hamiltonian describing the interaction of the cavity mode and squeezed vacuum reservoir is given as

$$\hat{H}_R(t) = i\sum_{i'} \lambda_{i'}(\hat{a}^{\dagger}\hat{c}_{i'}e^{i(\omega_a-\omega_{i'})t} - \hat{a}\hat{c}^{\dagger}_{i'}e^{-i(\omega_a-\omega_{i'})t}) + i\sum_{j'} \lambda_{j'}(\hat{b}^{\dagger}\hat{d}_{j'}e^{i(\omega_b-\omega_{j'})t} - \hat{b}\hat{d}^{\dagger}_{j'}e^{-i(\omega_b-\omega_{j'})t}). \tag{3}$$

Where $\hat{a}$ and $\hat{b}$ are the annihilation operator for the cavity modes with the frequency ω_a and ω_b, similarly ci', and dj' are annihilation operator for the reservoir modes having the frequency $\omega_{i'}$ and $\omega_{j'}$ but $\lambda_{i'}$ and $\lambda_{j'}$ are the coupling constants between the cavity modes and the reservoir modes. Here, we take the initial state of a single three-level atom considered to be,

$$|\psi(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle, \tag{4}$$

the corresponding initial density operator is given as;

$$\rho^{(0)} \;=\; \rho^{(0)}_{aa}|a\rangle\langle a| + \rho^{(0)}_{ac}|a\rangle\langle c| + \rho^{(0)}_{ca}|c\rangle\langle a| + \rho^{(0)}_{cc}|c\rangle\langle c|, \tag{5}$$

where $\rho^{(0)}_{aa}=|C_a(0)|^2$, $\rho^{(0)}_{cc}=|C_c(0)|^2$, are the probability for the atom to be in the upper and the lower levels at the initial time and

$$\rho^{(0)}_{ac} = C_a(0)C^*_c(0), \rho^{(0)}_{ca} = C_c(0)C^*_a(0), \tag{6}$$

represents the atomic coherence at the initial time. The master equation corresponding to Eq. 2 is

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) \;=\;& \frac{1}{2}A\rho^{(0)}_{aa}[2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}] \\
&+ \frac{1}{2}A\rho^{(0)}_{cc}[(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}] \\
&+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) \\
&+ \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}) \\
&- \frac{1}{2}A\rho^{(0)}_{ac}(2\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho}) \\
&- \frac{1}{2}A\rho^{(0)}_{ca}(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} - \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}),
\end{aligned} \tag{7}$$

The master equation resulted from the interaction of the cavity mode and a two mode squeezed vacuum reservoir is

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) \;=\;& -i[\hat{H}_R(t), \hat{\rho}(t)] + \frac{\kappa}{2}(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}) \\
&+ \frac{\kappa}{2}(2\hat{b}^{\dagger}\hat{\rho}\hat{b} - \hat{b}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^{\dagger}) \\
&+ \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) \\
&+ \frac{\kappa}{2}(N+1)(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}) \\
&+ \kappa M(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - \hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{b}) \\
&+ \kappa M(\hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}).
\end{aligned} \tag{8}$$

Finally, substituting Eq. (3) in to Eq. (8) we obtain the master equation of non-degenerate three level laser coupled to a two mode squeezed vacuum reservoir as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ &+ \frac{1}{2}[A\rho_{cc}^{(0)} + \kappa(N+1)](2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ &+ \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N)(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ &+ \frac{1}{2}\kappa N(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\ &+ \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M)(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}) \\ &+ \frac{\kappa M}{2}(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{b}) \\ &+ \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M)(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger) \\ &+ \frac{\kappa M}{2}(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger), \qquad (9)\end{aligned}$$

Eq. (9) indicates the stochastic master equation which contains all necessary information regarding the dynamics of the system involving the eect of the huge external environment. Moreover $A = \frac{2r_a g}{\gamma^2}$ represents the rate of injecting atoms which are initially prepared at the bottom level.

Employing the master equation, the time development of the c-number cavity modes variables, $\alpha(t)$ and $\beta(t)$, associated with the normal ordering, can be put in the form

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{\mu_a}{2}\langle\alpha\rangle - \frac{\rho_{ac}^{(0)}}{2}\langle\beta^*\rangle + f_\alpha, \qquad (10)$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{\mu_b}{2}\langle\beta\rangle + \frac{\rho_{ac}^{(0)}}{2}\langle\alpha^*\rangle + f_\beta, \qquad (11)$$

where

$$\mu_a = \kappa - A\rho_{aa}^{(0)}, \qquad (12)$$

$$\mu_b = \kappa + A\rho_{cc}^{(0)}, \qquad (13)$$

f_α and f_β are the pertinent noise forces associated with the fluctuation of the external environment, making use of Eqs. (10) and (11), the correlation properties of the noise forces can be readily put as

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0, \qquad (14)$$

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f_\beta^*(t')f_\beta^*(t)\rangle = 0, \qquad (15)$$

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = \langle f_\alpha^*(t')f_\alpha(t)\rangle = (A\rho_{aa}^{(0)} + \kappa N)\delta(t-t'), \qquad (16)$$

$$\langle f_\beta^*(t)f_\beta(t')\rangle = \langle f_\beta^*(t')f_\beta(t)\rangle = \kappa N\delta(t-t'), \qquad (17)$$

$$\langle f_\alpha(t)f_\beta(t')\rangle = \langle f_\alpha(t')f_\beta(t)\rangle = (\frac{1}{2}A\rho_{ac}^{(0)} + \kappa M)\delta(t-t') \qquad (18)$$

$$\langle f_\alpha(t)f_\beta^*(t')\rangle = \langle f_\alpha(t')f_\beta^*(t)\rangle = 0, \qquad (19)$$

$$\langle f_\beta^*(t')f_\alpha(t)\rangle = \langle f_\beta^* f_\alpha(t')(t)\rangle = 0, \qquad (20)$$

$$\langle f_\alpha(t)f_\alpha^*(t')\rangle = \langle f_\alpha(t')f_\alpha^*(t)\rangle = \kappa(N+1)\delta(t-t'). \qquad (21)$$

Continous Variable Entanglement Criteria

Under these section, we consider the Duan-Giedke-Cirac-Zoller, logarithm negativity, Hillery-Zubairy and Violationof Cauchy-Schwarz inequality criteria to quantify the degree of entanglement generated by the optical system.

A pair of particles is said to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents.

$$\hat{\rho} \neq \sum_j P_j\hat{\rho}_j^{(1)} \otimes \hat{\rho}_j^{(2)}, \qquad (22)$$

in which $P_j \geq 0$ and $\Sigma_j P_j = 1$ is set to ensure normalization of the combined density of state.

Duan-Giedke-Cirac-Zoller criterion

According to the criteria set by Duan et al a quantum state of the system is entangled provided that the sum of the variances of the two EPR-type operators $\hat{u}$ and $\hat{v}$ satisfies the condition [17],

$$(\Delta u)^2 + (\Delta v)^2 < 2, \qquad (23)$$

where,

$$\hat{u} = \hat{x}_a - \hat{x}_b, \qquad (24)$$

$$\hat{v} = \hat{p}_a + \hat{p}_b, \qquad (25)$$

$$\hat{x}_a = \frac{(\hat{a}^\dagger + \hat{a})}{\sqrt{2}}, \qquad (26)$$

$$\hat{x}_b = \frac{(\hat{b}^\dagger + \hat{b})}{\sqrt{2}}, \qquad (27)$$

$$\hat{p}_a = \frac{i(\hat{a}^\dagger - \hat{a})}{\sqrt{2}}, \qquad (28)$$

$$\hat{p}_b = \frac{i(\hat{b}^\dagger - \hat{b})}{\sqrt{2}}, \qquad (29)$$

being the quadrature operators for modes $\hat{a}$ and $\hat{b}$. Thus, the sum of the variances of $\hat{u}$ and $\hat{v}$ is easily found to be

$$(\Delta u)^2 + (\Delta v)^2 = 2\Big(1 - 2\langle\alpha(t)\beta(t)\rangle + \langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle\Big).$$

Now we seek to find the steady state solutions of the various parameters involved in Eq. (30). To this eect, assuming the initial states of the cavity modes to be in assuming the initial states of the cavity modes to be in the noise force at some time t does not aect the cavity modes variables at earlier times, it can be verified that

$$\langle\alpha^2\rangle = \langle\beta^2\rangle = \langle\alpha^*\beta\rangle = 0 \qquad (31)$$

Table 1: Degree of entanglement obtained at different value of η and squeezing parameter for κ = 0.8 and A = 100 from figure 2

r	Maximum degree of entanglement	Occurs at
$r = 0$	30.66%	$\eta = 0.17$
$r = 0.5$	70.25%	$\eta = 0.096$
$r = 0.75$	81.9%	$\eta = 0.07$

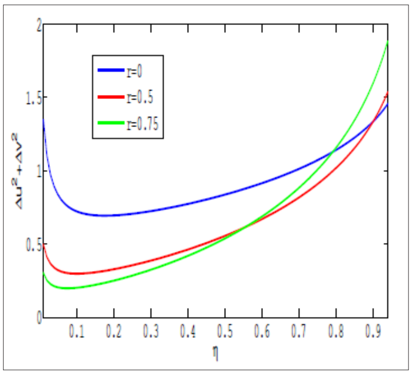

Figure 2: (color online). Plots of the sum of the variances of a pair of EPR-type operators $(\Delta u)^2 + (\Delta v)^2$ of the cavity radiation at steady state of [Eq.(30)] versus η for A=100, κ = 0.8, different values of r.

From this figure (2) we can understand that the light produced by the system under consideration is entangled state for all values of atomic coherence under consideration and entanglement criteria given by Eq.(23) is satisfied. Moreover, the entanglement of the cavity modes increases as the parameter r in the squeezed vacuum reservoir has increased and the two-mode squeezed vacuum reservoir considerably increases the amount of two-mode entanglement in the cavity for relatively small values of η. Moreover,from table 1 we can easily understand that the maximum degree of entanglement obtained for maximum value of the squeezing parameter at small value of the atomic coherence. In general better degree of entanglement is obtained in the presence of squeezing parameter than vacuum reservoir as our result.

$$\langle\alpha^*(t)\alpha(t)\rangle = \frac{\kappa A(1-\eta)(4\kappa+3A\eta+A)}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)} + \frac{[2\kappa(2\kappa+2A\eta+A)A^2(1+\eta)]2\kappa N}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)} - \frac{[A\sqrt{1-\eta^2}(2\kappa+A\eta+A)]2\kappa M}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)}, \quad (32)$$

Furthermore, in figure (3) we plotted the entanglement of the two mode light versus η and squeeze parameter r, it is easy to see from this plot that the squeezing increases with r and decrease with η in general. In order to see clearly the eect of the two mode squeezed vacuum reservoir on the degree of the squeezing of the two mode light generated by the laser system, this figure indicate that the two mode squeezed vacuum reser

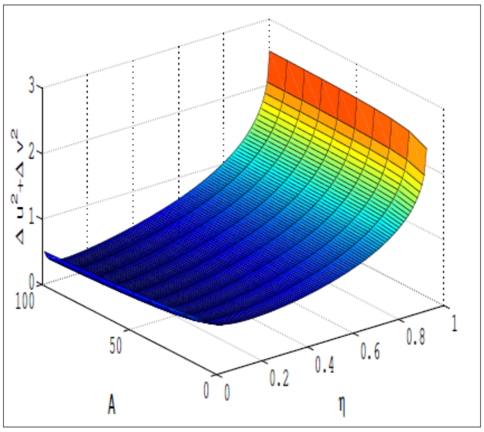

Figure 3: (color online). Plots of the sum of the variances of a pair of EPR-type operators $(\Delta u)^2+(\Delta v)^2$ of the cavity radiation at steady state of [Eq.(30)] versus η and A, for κ = 0.8, r = 0.5

voir considerably increases the amount of two mode entanglement in the cavity for relatively for small values of η and the condition assigned for the entangled state Eq.(23) is satisfied.

similarly, we get

$$\langle\beta^*(t)\beta(t)\rangle = \frac{[\kappa A^2(1-\eta^2)+(A\sqrt{1-\eta^2})(2\kappa+A\eta-A)]2\kappa M}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)} + \frac{[2\kappa(2\kappa+2A\eta-A)+A^2(1-\eta)]2\kappa N}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)} \quad (33)$$

$$\langle\alpha(t)\beta(t)\rangle = \frac{[A\kappa\sqrt{1-\eta^2}(2\kappa+A\eta+A)+(2\kappa+A\eta)^2-A^2]2\kappa M}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)} + \frac{[A^2\sqrt{1-\eta^2}]2\kappa N}{4\kappa(\kappa+A\eta)(2\kappa+A\eta)}. \quad (34)$$

By inserting Eqs.(32)-(34) in to Eq.(30) we get,

$$(\Delta u)^2+(\Delta v)^2 = 2[1+\frac{\kappa A(1-\eta)(2\kappa+2A\eta+A)-2\kappa A^2\eta^2 N}{2[\kappa(\kappa+A\eta)(2\kappa+A\eta)]} + \frac{\kappa A\sqrt{1-\eta^2}(2\kappa+A\eta+A)}{2[\kappa(\kappa+A\eta)(2\kappa+A\eta)]} + \frac{\kappa[2\kappa+A\eta(2\kappa+A\eta)(N+M)+A^2(1+\sqrt{1-\eta^2})(N-M)]}{[\kappa(\kappa+A\eta)(2\kappa+A\eta)]}]. \quad (35)$$

The result obtained in Eq.(35) represents the steady state solution of $(\Delta u)^2 + (\Delta v)^2$ quantum optical system.

Logarithmic Negativity

According to these criteria the presence of entanglement for a two-mode continuous variables based on the negativity of the partial transposition where the negative partial transpose must be parallel with respect to entanglement monotone in order to obtain the degree of entanglement. The logarithmic negativity for a twomode state is defined as [18],

$$E_N = max[0, -log_2 V_s]. \quad (36)$$

The logarithmic negativity is combined with negative partial transpose in another case where V represents the smallest eigenvalue of the symplectic matrix [18]

$$V_s = \left(\frac{\zeta-\sqrt{(\zeta^2-4det\Gamma)}}{2}\right)^{\frac{1}{2}}, \quad (37)$$

The entanglement is achieved when E_N is positive within the region of the lowest eigenvalue of co-variance matrix $V < 1$ [21,18]

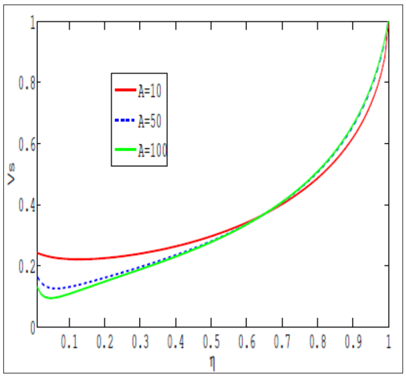

Figure 4: (color online). Plots of smallest eigen value of the symplectic matrix V [Eq.(37)] versus η for $r = 0.5$, $\kappa = 0.8$ and for different values of linear gain coecient

Figure (4) clearly shows that for smaller rate of atomic injection the maximum degree of entanglement prefers more number of atoms initially prepared in the lower energy state. However, for large rate of atomic injection entangled light is produced when atoms are initially prepared nearly closer to the maximum atomic coherence. For instance, according to this criteria the maximum degree of entanglement occurs at A = 100,

Table 2: Degree of entanglement obtained at different value of η and linear gain cocient for $\kappa = 0.8$ and $r = 0.5$ as shown on figure 4.

A	Maximum degree of entanglement	Occurs at
$A = 10$	78.2%	$\eta = 0.13$
$A = 50$	88%	$\eta = 0.06$
$A = 100$	91%	$\eta = 0.05$

η = 0.05 is 91%, the entanglement is achieved when E_N is positive within the region of the lowest eigenvalue of co-variance matrix $V < 1$, where the invariant and co-variance matrices are respectively denoted as

$$\zeta = det\zeta_1 + det\zeta_2 - 2\zeta_{12}, \quad (38)$$

$$\Gamma = \begin{pmatrix} \zeta_1 & \zeta_{12} \\ \zeta_{12}^T & \zeta_2 \end{pmatrix} \quad (39)$$

in which ζ_1 and ζ_2 are the co-variance matrices describing each mode separately while 12 are the inter-modal correlations.

The elements of the matrix on Eq. (39) can be obtained from the relation [18]

$$\Gamma_{ij} = 1/2\langle\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i\rangle - \langle\hat{\sigma}_i\rangle\langle\hat{\sigma}_j\rangle, \quad (40)$$

in which $i, j = 1,2,3,4$ the quadrature operators are defined as $\hat{\sigma}_1 = \hat{a} + \hat{a}^\dagger, \hat{\sigma}_2 = i(\hat{a}^\dagger - \hat{a}), \hat{\sigma}_3 = \hat{b} + \hat{b}^\dagger,$ and $\hat{\sigma}_4 = i(\hat{b}^\dagger - \hat{b})$.

We can easily understand from figure (5) that $V_s < 1$ for all values of under consideration showing that the radiation cavity is entangled for all parameter, so it satisfies the condition predicted in the logarithm negativity which is, $V_s < 1$, the maximum degree of entanglement achieved from this figure is 91% at $\eta = 0.05$, $r = 0.48$.

$$\Gamma = \begin{pmatrix} \delta_{11} & 0 & \delta_{13} & 0 \\ 0 & \delta_{22} & 0 & \delta_{24} \\ \delta_{31} & 0 & \delta_{33} & 0 \\ 0 & \delta_{42} & 0 & \delta_{44} \end{pmatrix}. \quad (41)$$

where $\delta_{11} = \delta_{22} = 2(\langle\alpha^*\alpha\rangle + 1), \delta_{33} = \delta_{44} = 2(\langle\beta^*\beta\rangle + 1), \delta_{23} = \delta_{32} = 0,$ $\delta_{13} = \delta_{31} = 2\langle\alpha\beta\rangle$, $\delta_{14} = \delta_{41} = 0, \delta_{23} = \delta_{32} = 0, \delta_{24} = \delta_{42} = -2\langle\alpha\beta\rangle$ $= -2\langle\alpha^*\beta^*\rangle$ and its simplified form is given by Eq.(32)-(34) respectively. Next on account of Eq. (39) along with the definitions of Eq. (41), one can readily show that,

$$det\zeta_1 = \left(\delta_{11}\right)^2 = \left(\delta_{22}\right)^2, \quad (42)$$

$$det\zeta_2 = \left(\delta_{33}\right)^2 = \left(\delta_{44}\right)^2, \quad (43)$$

$$det\zeta_{12} = det\zeta_{12}^T = -\left(\delta_{13}\right)^2 = -\left(\delta_{31}\right)^2. \quad (44)$$

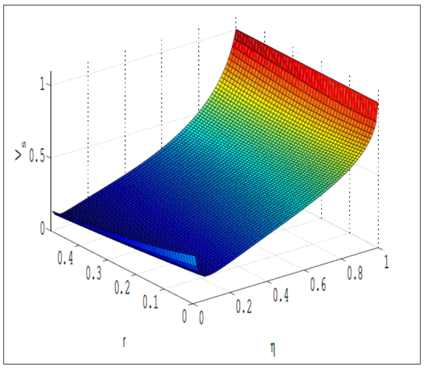

Figure 5: (color online). Plots of the smallest eigen value of the symplectic matrix V [Eq.(37)] versus η and r for $A = 100$, $\kappa = 0.8$

Using Eqs. (41) - (44) we can obtain the more reduced result as the following,

$$det\Gamma = \left[\sqrt{det\zeta_1 det\zeta_2} - \sqrt{(det\zeta_{12})^2}\right]^2. \quad (45)$$

The result presented in Eq.(37) along with Eqs.(38)- (45)represents the steady state expression of the smallest eigen value of the co-variance matrix V for the quantum optical system.

Hillery-Zubairy criterion

Entanglement is investigated in this study by the criteria conceived by Hillary and Zubairy according to these criteria two modes of the electromagnetic field with ˆa and ˆb annihilation operators, of composite state is said to be entangled if condition [19]

$$\langle n_a\rangle\langle n_b\rangle - \langle\hat{a}\hat{b}\rangle < 0 \quad (46)$$

is satisfied where $\hat{n}_a$ and $\hat{n}_b$ are the photon number operators corresponding to the involved cavity mode, whereas $\langle\hat{a}\hat{b}\rangle$ is the correlation of the cavity modes [19].

On the other hand, the second order correlation function for the mode corresponding to no time delay is given by

$$g^{(2)}(0) - 1 = \frac{\langle\alpha\beta\rangle^2}{\langle\alpha^*\alpha\rangle\langle\beta^*\beta\rangle} \quad (47)$$

is the equal-time second-order (intensity) correlation function which satifies the following condition,

$$g^{(2)}(0) \geq 2 \quad (48)$$

It has been shown that the Hillery-Zubairy criterion is another equivalent entanglement criterion with Cauchy- Stewart inequality when the inter-atomic interaction is not taken into account [22]. This is contrary to the other criterion and the result reported in in which the driven atomic coherence is used. The criteria can be rewritten as [16, 23].

$$H_z = \langle n_a \rangle \langle n_b \rangle - \langle \hat{a}\hat{b} \rangle^2 \quad (49)$$

in which the negativity of the parameter H_z is a clear indication of the existence of entanglement [24]. In terms of the c-number and zero mean Gaussian variables, we see that

$$H_z = \langle \alpha^* \alpha \rangle \langle \beta^* \beta \rangle - \langle \alpha\beta \rangle^2 \quad (50)$$

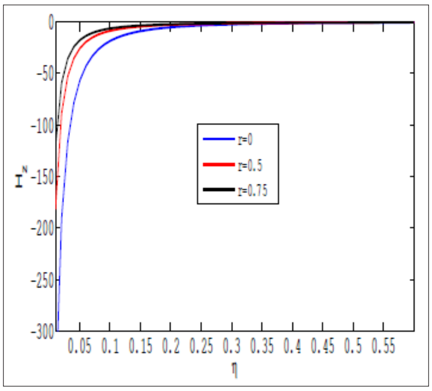

Figure 6: (color online). Plots of H_z of [Eq.(50)] versus η for A = 100, κ = 0.8 and different value of squeezing parameter.

As it can bee seen from figure 6 the parameter H_z is less than zero and becomes more negative by decreasing the squeezing parameter. Moreover, we observe that H_z closes to zero as the parameter r is increased indicating that the entanglement depletes with the squeezing parameter and the plot satisfies the condition assigned for entangled state by the criteria on Eq.(46).

Violation of Cauchy-Schwarz inequality (VCSI)

A system of two-mode cavity radiation is said to be entangled if it satisfies the Cauchy-Schwarz inequality in the form:

$$\langle \hat{a}^{\dagger 2}\hat{a}^2 \rangle \langle \hat{b}^{\dagger 2}\hat{b}^2 \rangle \geq \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle^2. \quad (51)$$

In this relation it is possible to study the non-classical photon number correlation at equal time using the following parameter [22]:

$$C_{ab} = \frac{|\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle|^2}{\langle \hat{a}^{\dagger 2}\hat{a}^2 \rangle \langle \hat{b}^{\dagger 2}\hat{b}^2 \rangle}. \quad (52)$$

in which the two-mode light is entangled when $C_{ab} \geq 1$ Since the operators are already put in the normal order, the photon number correlation can be expressed, in terms of the c-number zero mean Gaussian variables and at steady state, as [17, 21]:

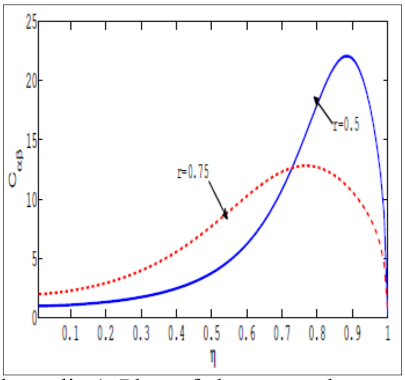

Figure 7: (color online). Plots of photon number correlation $C_{\alpha\beta}$ of [Eq.(57)] versus η for A = 100, κ = 0.8, and for different value of squeezing parameter.

$$\langle \alpha^* \alpha \beta^* \beta \rangle = \langle \alpha^* \alpha \rangle \langle \beta^* \beta \rangle + \langle \alpha\beta \rangle^2, \quad (53)$$

$$\langle \alpha^{*2} \alpha^2 \rangle = 2\langle \alpha^* \alpha \rangle^2, \quad (54)$$

$$\langle \beta^{*2} \beta^2 \rangle = 2\langle \beta^* \beta \rangle^2. \quad (55)$$

Now applying Eqs. (53)-(55)in to Eqs. (52) , we find that:

$$C_{\alpha\beta} = \frac{1}{4}\left[1 + \frac{\langle \alpha\beta \rangle^2}{\langle \alpha^* \alpha \rangle \langle \beta^* \beta \rangle}\right]^2.$$

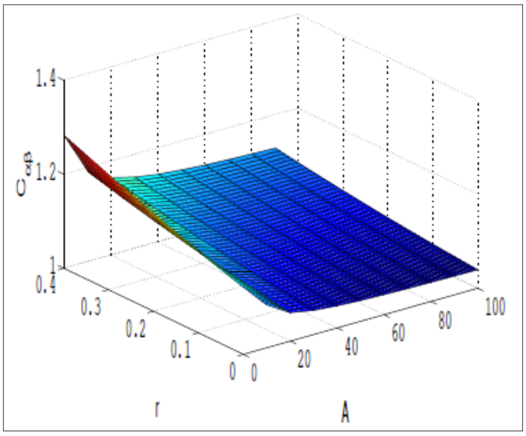

Figure 8: (color online). Plots of photon number correlation $C_{\alpha\beta}$ of [Eq.(57)] v ersus A and r, for x = 0.1, κ = 0.8.

It can be verified as,

$$C_{\alpha\beta} = \frac{1}{2}[g^{(2)}(0)]^2, \quad (57)$$

where

$$g^{(2)}(0) = 1 + \frac{\langle \alpha\beta \rangle^2}{\langle \alpha^* \alpha \rangle \langle \beta^* \beta \rangle} \quad (58)$$

is the equal-time second-order (intensity) correlation function. As it can be seen from figure (7) the photon number correlation increases with the squeezing parameter for some value of η and decreases for other value, it is clearly shown that the photon number correlation increases with the squeezing parameter for small value of and decreases for large value. In this case for $0 \leq \eta \leq 0.7$ the photon number correlation increase as the squeezing parameter increases but for $0.7 < \eta \leq 1$ the photon number correlation start to decrease as the squeezing parameter increases, this implies that the photon number correlation falls below for $\eta = 1$ which indicate the fact that the entanglement vanishes in the absence of the atomic coherence and it implies that that the correlation between the photon numbers tends to be maximum in the regions where the entanglement is minimum for $0 \leq \eta \leq 0.7$, therefore the correlation between the photon numbers tends to be minimum in the regions where the entanglement are maximum for $0.7 < \eta \leq 1$ in this case. Moreover, it is clearly shown in figure (8) that the photon number correlation, $C_{\alpha\beta} \geq 1$, for all values of parameters under consideration. This indicates that the non-degenerate three-level cascade laser coupled to two-mode squeezed vacuum reservoir is a source of entangled light, according to the Cauchy-Schwartz inequality and HZ criteria. It is also observed that the criterion does not include the case for which entanglement is weak when the procedure following from the logarithmic negativity and DGCZ criteria are applied.

From figure 9 we can understand that how second order correlation is vary with the linear gain coecient for different values of squeezing parameter, as it is clearly observed from the figure the second order correlation function increases for small value of the linear gain co-effcient and decreases for large value of linear gain co-effcient also from these plot we can see that second order correlation function increases as the squeezing parameter increases.

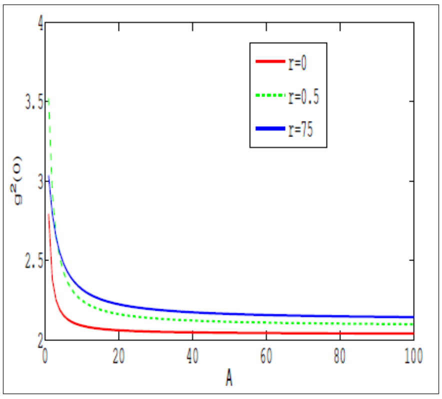

Figure 9: (color online). Plots of second order correlation function g2(0) of [Eq.(58)] versus A , for $x = 0.1$, $\kappa = 0.8$ and different values of the squeezing parameter.

From figure 10 we see that the eect of atomic coherence and linear gain coecient on the second order correlation function. Hence, we can see that the correlation of the photon number increases with increasing atomic coherence under consideration. We also found that for η very close to 1 the correlation of the photon number would be significantly small. Furthermore, it clearly shows the correlation of the photon number increases as linear gain coecient decreases. However, we found that the degree of the entanglement increases with the linear gain coecient, so from these results the correlation between the photon numbers get to be minimum in the region where the entanglement is maximum Generally, from figure (9), and (10) the condition assigned for second order correlation function $g^2(0) \geq 2$ on Eq.(48) is satisfied.

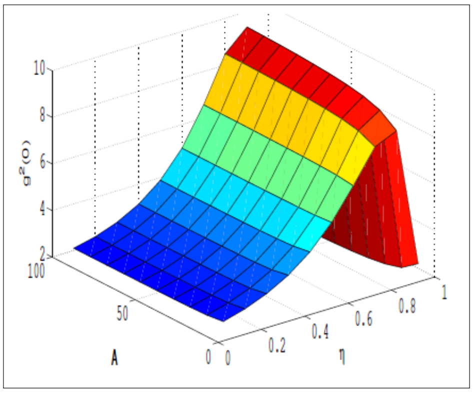

Figure 10: (color online). Plots of second order correlation function $g^2(0)$ of [Eq.(58)] versus A and η, for $r = 0.5$, $\kappa = 0.8$.

Photon antibunching

A photon antibunching phenomenon happens when the statistics of photons is scattered by passing time. It corresponds to fewer photon pairs detecting closer together in time. The correlation of scattered photons is studied via the second-order correlation function of photo detection with respect to time, based on a photo detection experiment, for a coherent state, $g^2(\tau) = 1$ represents the highly correlated state [25, 26, 20]. In this state, the probability of joint detection coincides with the probabilities of independent detection [20]. On the other hand, $g^2(\tau) = 0$ when the time delay approaches infinity, which means the joint probability of detecting the second photon decreases with time delay. Thus, the situation $g^2(\tau) < g^2(0)$ is identified as photon bunching which means two photons tend to be detected simultaneously or after a short time delay [20]. If $g^2(\tau) > g^2(0)$, the joint probability of detecting the second photon increases $g^2(\tau) \to 1$ with time delay which is known as photon antipunching Here, g^2 for τ approach to 1 and $g^2(\tau) < 1$ implies the increased probability of detecting a second photon after a finite time delay, τ This contradiction is the result of the quantum nature of light [26]. Thus, photon anti-bunching is one of the methods to describe the entanglement. A field is said to be entangled if the inequality $g^2(\tau) > g^2(0)$ is satisfied [20]. For the coherent state, $g^2(\tau) = 1$ represents a classical state. However, for a non-classical state, we have $g^2(\tau) < 1$ which corresponds to the photon antibunching phenomenon occurrence. When $g^2(0) < 1$ and $g^2(\tau) > g^2(0)$, implying the presence of entanglement [20]. Unfortunately, it has been verified that $g^2(0) > 1$ for a non-degenerate three-level cascade laser while it is exhibiting the entanglement in some condition [25- 32].

Conclusion

In this paper, we quantify the the degree of entanglement generated in non-degenerate three-level laser coupled to a two-mode squeezed vacuum reservoir by using different inseparability criteria. First, we determined the master equation in the good-cavity limit, linear and adiabatic approximation schemes. Applying the resulting master equation, we have derived equations of evolution of the cavity mode variables, with the aid of these equations, entanglement of the two-mode cavity light was discussed and it is found that the the squeezing parameter of the squeezed vacuum reservoir produces a significant degree of entanglement. The maximum achievable degree of entanglement of each case increases with minimum atomic coherence. As a result, further increment of the squeezing parameter leads to the maximum degree of entanglement at the maximum atomic coherence when the rate of atomic injection into the laser cavity is relatively large. Although the degree of entanglement at the maximum atomic coherence is enhanced, the observed situation immediately reverses at the minimum atomic coherence. In other words, the inter modal correlation between sub-modes of the squeezed vacuum reservoir comes into play mostly when large number of photons are available in the laser cavity. Contrary to this, the entanglement properties of the cavity radiation decay fast at the minimum atomic coherence even if the system is coupled with the squeezed vacuum reservoir. Moreover, we found that the degree of entanglement obtained by using the logarithm negativity as 77.9% for $A = 10$ at $\eta = 0.12$, 88% for $A = 50$ at $\eta = 0.6$, and 91% for $A = 100$, $r = 0.5$ at $\eta = 0.05$ from figure (4), from these result we can understand that maximum degree of entanglement is achieved from smaller value of atomic coherence and maximum value of linear gain coecient , similarly the Hillary-Zubairy demonstrated the entanglement property of the cavity radiation like the logarithm negativity. However, in Hillary-Zubairy criteria there is no lower limit on the value of the parameter H_z, as a result we could not exactly know the degree of entanglement generated in this criteria, but we can clearly identify the weaker and stronger entangled light, that means according to this criteria the strong entangled light occurred at a more negative value of the parameter H_z. Moreover, using Cauchy-Schwartz inequality we found that the photon number correlation increases with the squeezing parameter for some values of η and decreases for other values of η until it vanishes, from these we can clearly understand that the correlation between the photon numbers tends to be minimum in the regions where the entanglement are maximum. In general, we have found that there is a significant entanglement between the states of the light generated in the cavity of a non-degenerate three-level cascade laser due to the strong correlation of the light emitted, when the atom decays from the top level to the bottom level via the intermediate level. The results we found indicate that the two cavity modes are strongly entangled.

Data Availability

The data is included in the manuscript.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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