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On Finding Determinantal Equations Involving the Coordinates of the Object Point, Point of Incidence, Point of Observation and the Image Point in Cases of Reflection and Refraction

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ABSTRACT

This paper is concerned with the procedure of finding determinantal equations, each of which involves the coordinates of the object point, point of incidence, point of observation, and the image point in cases of reflection and refraction of light. An algorithm has been presented for the said purpose and it has been subsequently employed for the generation of relevant determinantal equations by considering some particular cases of reflection and refraction, based on which the readers will be able to reach the desired goal for all other remaining cases of reflection and refraction. Overall, two sets of such determinantal equations have been found to prevail in the present study. One resulted from handling the following cases: (i) Reflection at a plane reflecting surface, (ii) Refraction at a plane surface of discontinuity regardless of whether light moves from a rarer medium to a denser medium or from a denser medium to a rarer medium. The second one has been found while going through the following cases: (i) Reflection at a concave spherical reflecting surface, (ii) Reflection at a convex spherical reflecting surface, (iii) Refraction at a concave spherical surface of discontinuity regardless of whether light passes from a rarer medium to a denser medium or from a denser medium to a rarer medium, (iv) Refraction at a convex spherical surface of discontinuity regardless of whether light moves from a rarer medium to a denser medium or from a denser medium to a rarer medium.

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Introduction

The generalized vectorial laws of reflection and refraction [1] have been applied earlier to solve a wide variety of problems in Geometrical optics with the development of interesting physical insights to a lot of optical phenomena, a sum up of which is available in [2]. Recently, the generalized vectorial laws of reflection and refraction [1] have been applied in [3,4] to solve two novel problems falling within the purview of Geometrical optics. In [3], the said laws of reflection and refraction have been employed to develop novel determinantal equations involving the coordinates of the source point, point of incidence, and the point of observation in cases of reflection and refraction of light. The generalized vectorial laws of reflection and refraction have also been used in [4] for the generation of novel determinantal equations involving the coordinates of the object point, image point, and the point of incidence in cases of reflection and refraction along with offering novel treatments for the derivation of a few well known results in Geometrical optics. Working in line with the earlier works [3,4] this paper reports on the presentation of a novel algorithm for the generation of determinantal equations involving the coordinates of the object point, point of incidence, point of observation, and the image point on the basis of the generalized vectorial laws of reflection and refraction. The proposed algorithm has been subsequently applied to illustrate the procedure of generation of relevant determinantal equations with reference to a few typical cases of reflection and refraction by providing hints for all other remaining cases so that interested readers may be able to reach the desired goal quite easily. It has been found that the generalized vectorial laws of reflection and refraction lead to the development of two sets of novel determinantal equations, each one involving the coordinates of the object point, point of incidence, point of observation, and the image point in cases of reflection (refraction) of light at plane and spherical reflecting surfaces (plane and spherical surfaces of discontinuity).

The algorithm presented will be helpful for the generation of relevant determinantal equations. Furthermore, in addition to novelty and originality, the present work increases the range of applicability of the generalized vectorial laws of reflection and refraction and it must be useful from the view point of academic interest as well.

Algorithm for Finding Determinantal Equations Involving Coordinates of the Object Point, Point of Incidence, Point of Observation, and the Image Point

Step 1: With reference to a right-handed system of coordinates, denote the equations of the plane reflecting surface (plane surface of discontinuity) as well as the equation of the curved spherical reflecting surface (surface of discontinuity) by $ax+by+cz+d=0$;or,$\varphi(x,y,z)= ax+by+cz+d=0$ and $x^2+ y^2+ z^2=$

a^2;or,$\varphi(x,y,z)= x^2+ y^2+ z^2- a^2=0$ respectively, as the case may be.

Step 2: With reference to the same right-handed system of coordinates chosen in Step 1, denote the coordinates of the object point, point of incidence, point of observation, and the image point by (x_1,y_1,z_1), (x_3,y_3,z_3), (x_2,y_2,z_2),and (x_4,y_4,z_4) respectively.

Step 3: Considering the relevant Figure for reflection (refraction) and making use of the coordinates of the object point, point of incidence, point of observation, the image point and the equation of the plane reflecting surface (surface of discontinuity) or the curved spherical reflecting surface (surface of discontinuity), as the case may be, obtain the expressions for the unit vectors **i, r** (or **R**), and **n** along the directions of the incident ray, reflected (or refracted) ray, and the positive normal to the reflecting (refracting) surface at the point of incidence respectively in terms of the rectangular unit vectors **I**, **J**, and **K**. Also, in finding the expression for **r** (or **R**), care must be taken to see that such an expression for **r** (or **R**) must contain the coordinates of the point of observation and the image point.

Step 4: Substitute the expressions for **i**, **r** (or **R**), and **n** obtained in Step 3 in the generalized vectorial law of reflection (refraction), Viz. $\mathbf{n} \times \mathbf{i}=\mathbf{n} \times \mathbf{r}$ (or $\mathbf{n} \times \mathbf{i}=\mu(\mathbf{n} \times \mathbf{R})$, or $\mathbf{n} \times \mathbf{i} = \frac{1}{\mu}(\mathbf{n} \times \mathbf{R})$).

Step 5: Comparing the corresponding coefficients of **I**, **J**, and **K** on both sides of the equation obtained in Step 4, obtain three sets of equation thereafter.

Step 6: By eliminating a common term from each pair of equations obtained in Step 5, develop three more equations, each of which involves the coordinates of the object point, point of incidence, point of observation, and the image point.

Step 7: Then verify that each of the equations obtained in Step 6 leads to the same determinantal equation to be arrived at.

Application of the Algorithm for the Generation of Determinantal Equations

Let us now apply the above algorithm to illustrate the procedure of generation of the determinantal equations involving coordinates of the object point, point of incidence, point of observation, and the image point in cases of reflection and refraction of light. On account of space limitation, discussion will be kept limited to the following four typical cases, leaving the generation of determinantal equations for all other remaining cases to the interested readers.

Case 1: For Reflection at a Plane Reflecting Surface

Let us consider Figure. 1 for the generation of the relevant determinantal equation for this case by applying the above algorithm.

Following Step 1 of the algorithm, with reference to a right-handed system of coordinates (not shown in Figure. 1), we first denote the equation of the plane reflecting surface by ax+by+cz+d=0;or,$\varphi(x,y,z)=$ ax+by+cz+d=0, provided a, b, c are not simultaneously equal to zero and then following Step 2, the coordinates of the object point A, point of incidence B, point of observation D, and the image point E are assigned as(x_1,y_1,z_1),(x_3,y_3,z_3), (x_2,y_2,z_2),and (x_4,y_4,z_4) respectively.

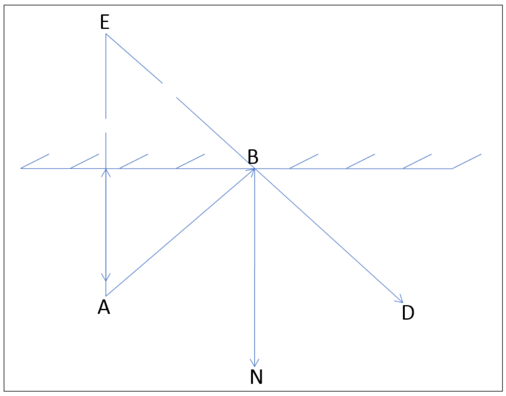

Figure 1: Diagram showing reflection of light at a plane mirror in which A is the object point, B is the point of incidence, D is the point of observation, E is the image point, BN being the positive normal to the reflector at the point of incidence

Following Step 3, we then obtain the following expressions for **i**, **r**, and **n** by considering Figure 1.

$$\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|} = \frac{[(x_3 - x_1)\,\mathbf{I} + (y_3 - y_1)\,\mathbf{J} + (z_3 - z_1)\,\mathbf{K}]}{[U_1]},$$

where $U_1 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}$.

$$\mathbf{r} = \frac{\mathbf{ED}}{|\mathbf{ED}|} = \frac{[(x_2 - x_4)\,\mathbf{I} + (y_2 - y_4)\,\mathbf{J} + (z_2 - z_4)\,\mathbf{K}]}{[U_2]},$$

where $U_2 = \sqrt{(x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2}$.

$$\mathbf{n} = \frac{[\text{grad}\,\varphi(x,y,z)]_{(x_3,y_3,z_3)}}{|[\text{grad}\,\varphi(x,y,z)]_{(x_3,y_3,z_3)}|} = \frac{a\,\mathbf{I} + b\,\mathbf{J} + c\,\mathbf{K}}{W},$$

where $W = \sqrt{a^2 + b^2 + c^2}$.

Substituting the above expressions for **i**, **r**, and **n** in the generalized vectorial law of reflection (viz. $\mathbf{n} \times \mathbf{i}=\mathbf{n} \times \mathbf{r}$) as per Step 4 of the algorithm, we then obtain the following equation.

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{a}{W} & \frac{b}{W} & \frac{c}{W} \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{a}{W} & \frac{b}{W} & \frac{c}{W} \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

or

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a & b & c \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a & b & c \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

Comparison of the corresponding coefficients of **I**, **J**, and **K** on both sides of the above relation is then essential as per Step 5 and that yields the following three sets of equations.

$$\frac{U_1}{U_2} = \frac{\{b(z_3 - z_1) - c(y_3 - y_1)\}}{\{b(z_2 - z_4) - c(y_2 - y_4)\}} \tag{1}$$

$$\frac{U_1}{U_2} = \frac{\{a(z_3 - z_1) - c(x_3 - x_1)\}}{\{a(z_2 - z_4) - c(x_2 - x_4)\}} \tag{2}$$

$$\frac{U_1}{U_2} = \frac{\{a(y_3 - y_1) - b(x_3 - x_1)\}}{\{a(y_2 - y_4) - b(x_2 - x_4)\}} \tag{3}$$

Following Step 6, it is then required to eliminate the common term $\frac{U_1}{U_2}$ from each pair of the above three equations.

Thus eliminating the term $\frac{U_1}{U_2}$ from equations (1) and (2) we get,

$$\frac{\{b(z_3 - z_1) - c(y_3 - y_1)\}}{\{b(z_2 - z_4) - c(y_2 - y_4)\}} = \frac{\{a(z_3 - z_1) - c(x_3 - x_1)\}}{\{a(z_2 - z_4) - c(x_2 - x_4)\}}$$

Since in this case, c≠0, this equation reduces to

$$a[(y_2 - y_4)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_4)] - b[(x_2 - x_4)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_4)] + c[(x_2 - x_4)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_4)] = 0 \quad (4)$$

Again, eliminating the term $\frac{U_1}{U_2}$ from equations (2) and (3), we have,

$$\frac{\{a(z_3 - z_1) - c(x_3 - x_1)\}}{\{a(z_2 - z_4) - c(x_2 - x_4)\}} = \frac{\{a(y_3 - y_1) - b(x_3 - x_1)\}}{\{a(y_2 - y_4) - b(x_2 - x_4)\}}$$

Since in this case, a≠0, this equation reduces to

$$a[(y_2 - y_4)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_4)] - b[(x_2 - x_4)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_4)] + c[(x_2 - x_4)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_4)] = 0 \quad (5)$$

Also, eliminating the term $\frac{U_1}{U_2}$ from equations (3) and (1), we get,

$$\frac{\{a(y_3 - y_1) - b(x_3 - x_1)\}}{\{a(y_2 - y_4) - b(x_2 - x_4)\}} = \frac{\{b(z_3 - z_1) - c(y_3 - y_1)\}}{\{b(z_2 - z_4) - c(y_2 - y_4)\}}$$

Since in this case, b≠0, this equation reduces to

$$a[(y_2 - y_4)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_4)] - b[(x_2 - x_4)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_4)] + c[(x_2 - x_4)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_4)] = 0 \quad (6)$$

Following Step 7, it is then easy to see that each of the equations (4), (5), and (6) ultimately leads to the following novel determinantal equation.

$$\begin{vmatrix} a & b & c \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \end{vmatrix} = 0 \quad (7)$$

Case 2: For Refraction of Light at a Plane Surface of Discontinuity as it passes From a Rarer Medium to a Denser Medium
In order to generate the determinantal equation for this case we are now to consider Figure. 2. As before, following Step 1 of the algorithm, by choosing a right-handed system of coordinates (not shown in Figure. 2), we first denote the equation of the plane surface of discontinuity by ax+by+cz+d=0;or,φ(x,y,z)= ax+by+cz+d=0, provided a, b, c are not simultaneously equal to zero and thereafter, following Step 2, the coordinates of the object point A, point of incidence B, point of observation D, and the image point E are assigned as(x_1,y_1,z_1), (x_3,y_3,z_3),(x_2,y_2,z_2),and (x_4,y_4,z_4) respectively.

The following expressions for **i**, **R**, and **n** are then obtained by considering Figure 2 as per Step 3.

$$\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|} = \frac{[(x_3 - x_1)\,\mathbf{I} + (y_3 - y_1)\,\mathbf{J} + (z_3 - z_1)\,\mathbf{K}]}{[U_1]},$$

where $U_1 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}$.

$$\mathbf{R} = \frac{\mathbf{ED}}{|\mathbf{ED}|} = \frac{[(x_2 - x_4)\,\mathbf{I} + (y_2 - y_4)\,\mathbf{J} + (z_2 - z_4)\,\mathbf{K}]}{[U_2]},$$

where $U_2 = \sqrt{(x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2}$.

$$\mathbf{n} = \frac{[\text{grad } \varphi(x,y,z)]_{(x_3,y_3,z_3)}}{|[\text{grad } \varphi(x,y,z)]_{(x_3,y_3,z_3)}|} = \frac{a\,\mathbf{I} + b\,\mathbf{J} + c\,\mathbf{K}}{W},$$

where $W = \sqrt{a^2 + b^2 + c^2}$.

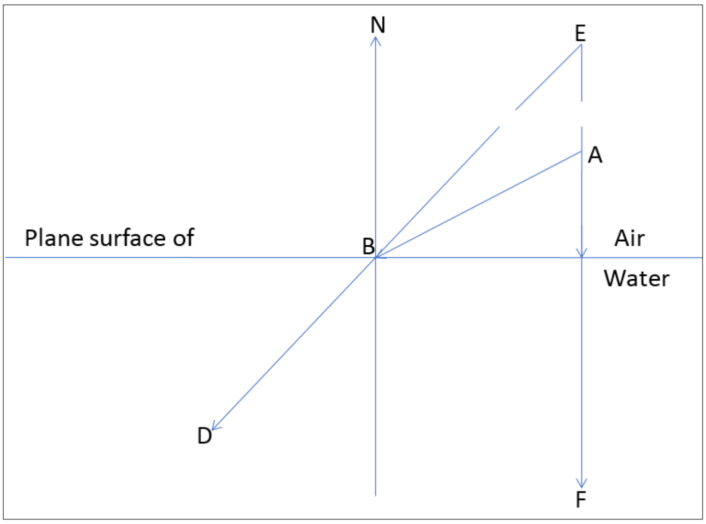

Figure 2: Diagram showing refraction of light at a plane surface of discontinuity as it passes from air to water in which A is the object point, B is the point of incidence, D is the point of observation, E is the image point, BN being the positive normal to the plane surface of discontinuity at the point of incidence

By Step 4, we are then to substitute the above expressions for **i**, **R**, and **n** in the generalized vectorial law of refraction (viz. $\mathbf{n} \times \mathbf{i} = \mu (\mathbf{n} \times \mathbf{R})$, where μ is the refractive index of water with respect to air) to obtain the following equation.

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{a}{W} & \frac{b}{W} & \frac{c}{W} \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \mu \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{a}{W} & \frac{b}{W} & \frac{c}{W} \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

or

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a & b & c \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \mu \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a & b & c \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

Following Step 5, we are now to compare the corresponding coefficients of **I**, **J**, and **K** on both sides of the above relation to get the following three sets of equations.

$$\frac{\mu U_1}{U_2} = \frac{\{b(z_3 - z_1) - c(y_3 - y_1)\}}{\{b(z_2 - z_4) - c(y_2 - y_4)\}} \tag{8}$$

$$\frac{\mu U_1}{U_2} = \frac{\{a(z_3 - z_1) - c(x_3 - x_1)\}}{\{a(z_2 - z_4) - c(x_2 - x_4)\}} \tag{9}$$

$$\frac{\mu U_1}{U_2} = \frac{\{a(y_3 - y_1) - b(x_3 - x_1)\}}{\{a(y_2 - y_4) - b(x_2 - x_4)\}} \tag{10}$$

Following Step 6, we are then to eliminate the common term $\frac{\mu U_1}{U_2}$ from each pair of the above three equations to obtain the following three equations.

$$\frac{\{b(z_3 - z_1) - c(y_3 - y_1)\}}{\{b(z_2 - z_4) - c(y_2 - y_4)\}} = \frac{\{a(z_3 - z_1) - c(x_3 - x_1)\}}{\{a(z_2 - z_4) - c(x_2 - x_4)\}} \tag{11}$$

$$\frac{\{a(z_3 - z_1) - c(x_3 - x_1)\}}{\{a(z_2 - z_4) - c(x_2 - x_4)\}} = \frac{\{a(y_3 - y_1) - b(x_3 - x_1)\}}{\{a(y_2 - y_4) - b(x_2 - x_4)\}} \tag{12}$$

$$\frac{\{a(y_3 - y_1) - b(x_3 - x_1)\}}{\{a(y_2 - y_4) - b(x_2 - x_4)\}} = \frac{\{b(z_3 - z_1) - c(y_3 - y_1)\}}{\{b(z_2 - z_4) - c(y_2 - y_4)\}} \tag{13}$$

In accordance with Step 7, it can now be readily verified that, each of the above three equations (11), (12) and (13) is equivalent to the following determinantal equation for this particular case of refraction since here a, b, c are not simultaneously zero.

$$\begin{vmatrix} a & b & c \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \end{vmatrix} = 0 \tag{14}$$

It is now interesting to see that the determinantal equation (14) in this case is exactly identical with the determinantal equation (7) obtained earlier in Case 1 for dealing with the case of reflection at a plane reflecting surface.

Note 1: Following the same procedure as followed in Case 2 above by applying the algorithm presented earlier, the readers could easily verify that the same determinantal equation (7) or (14) will also result when considering the case of refraction at a plane surface of discontinuity as light passes from a denser medium to a rarer medium having in mind that the generalized vectorial law of refraction to be used for such a case is $\mathbf{n} \times \mathbf{i} = \frac{1}{\mu}(\mathbf{n} \times \mathbf{R})$, where μ is the refractive index of the denser medium with respect to the rarer medium.

Case 3: For Reflection at a Convex Spherical Reflecting Surface

In order to deal with this case, we are to consider Figure. 3.

As per Step 1 of the algorithm, let us first assume that, with reference to a right-handed system of coordinates (not shown in Figure 3), the equation of the convex spherical reflecting surface be given by $x^2 + y^2 + z^2 = a^2$;or, $\varphi(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$ and then following Step 2, let us also assume that the coordinates of the object point A, point of incidence B, point of observation D, and the image point E be (x_1,y_1,z_1), (x_3,y_3,z_3), (x_2,y_2,z_2),and (x_4,y_4,z_4) respectively.

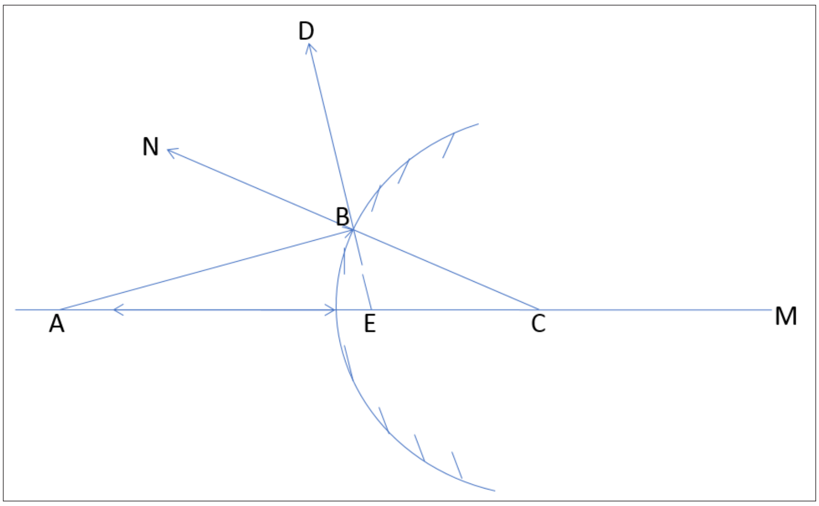

Figure 3: Diagram showing reflection of light at a convex spherical reflecting surface in which A is the object point, B is the point of incidence, D is the point of observation, E is the image point, C is the center of curvature of the convex spherical mirror with AM as its principal axis, CN being the positive normal to the reflector at the point of incidence

Following Step 3 of the algorithm, the following expressions for **i**, **r**, and **n** are then obtained by considering Figure 3.

$$\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|} = \frac{[(x_3 - x_1)\,\mathbf{I} + (y_3 - y_1)\,\mathbf{J} + (z_3 - z_1)\,\mathbf{K}]}{[U_1]},$$

where $U_1 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}$.

$$\mathbf{r} = \frac{\mathbf{ED}}{|\mathbf{ED}|} = \frac{[(x_2 - x_4)\,\mathbf{I} + (y_2 - y_4)\,\mathbf{J} + (z_2 - z_4)\,\mathbf{K}]}{[U_2]},$$

where $U_2 = \sqrt{(x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2}$.

$$\mathbf{n} = \frac{[\text{grad}\,\varphi(x,y,z)]_{(x_3,y_3,z_3)}}{|\,[\text{grad}\,\varphi(x,y,z)]_{(x_3,y_3,z_3)}\,|} = \frac{1}{a}\,(x_3\,\mathbf{I} + y_3\,\mathbf{J} + z_3\,\mathbf{K})$$

As per Step 4 of the algorithm, it is then essential to substitute the above expressions for **i**, **r**, and **n** in the generalized vectorial law of reflection (viz. **n** ×**i**=**n** ×**r**) so as to obtain the following equation.

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{x_3}{a} & \frac{y_3}{a} & \frac{z_3}{a} \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{x_3}{a} & \frac{y_3}{a} & \frac{z_3}{a} \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

or

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ x_3 & y_3 & z_3 \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ x_3 & y_3 & z_3 \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

Following Step 5 of the algorithm, we then compare the corresponding coefficients of **I**, **J**, and **K** on both sides of the above relation and obtain the following three sets of equations.

$$\frac{U_1}{U_2} = \frac{\{y_3\,(z_3 - z_1) - z_3\,(y_3 - y_1)\}}{\{y_3\,(z_2 - z_4) - z_3\,(y_2 - y_4)\}} \tag{15}$$

$$\frac{U_1}{U_2} = \frac{\{x_3\,(z_3 - z_1) - z_3\,(x_3 - x_1)\}}{\{x_3\,(z_2 - z_4) - z_3\,(x_2 - x_4)\}} \tag{16}$$

$$\frac{U_1}{U_2} = \frac{\{x_3\,(y_3 - y_1) - y_3\,(x_3 - x_1)\}}{\{x_3\,(y_2 - y_4) - y_3\,(x_2 - x_4)\}} \tag{17}$$

Eliminating the common term $\frac{U_1}{U_2}$ from each pair of the above three equations, as per Step 6 of the algorithm, we then obtain the following three equations.

$$\frac{\{y_3\,(z_3 - z_1) - z_3\,(y_3 - y_1)\}}{\{y_3\,(z_2 - z_4) - z_3\,(y_2 - y_4)\}} = \frac{\{x_3\,(z_3 - z_1) - z_3\,(x_3 - x_1)\}}{\{x_3\,(z_2 - z_4) - z_3\,(x_2 - x_4)\}} \tag{18}$$

$$\frac{\{x_3\,(z_3 - z_1) - z_3\,(x_3 - x_1)\}}{\{x_3\,(z_2 - z_4) - z_3\,(x_2 - x_4)\}} = \frac{\{x_3\,(y_3 - y_1) - y_3\,(x_3 - x_1)\}}{\{x_3\,(y_2 - y_4) - y_3\,(x_2 - x_4)\}} \tag{19}$$

$$\frac{\{x_3\,(y_3 - y_1) - y_3\,(x_3 - x_1)\}}{\{x_3\,(y_2 - y_4) - y_3\,(x_2 - x_4)\}} = \frac{\{y_3\,(z_3 - z_1) - z_3\,(y_3 - y_1)\}}{\{y_3\,(z_2 - z_4) - z_3\,(y_2 - y_4)\}} \tag{20}$$

Following Step 7, it is now easy to see that, each of the above three equations (18), (19) and (20) leads to the following determinantal equation for this particular case of reflection, provided that the point of incidence is not the origin of the coordinate system so that x_3, y_3, z_3 are not simultaneously zero.

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \end{vmatrix} = 0 \qquad (21)$$

Case 4: For Refraction at a Concave Spherical Surface of Discontinuity as Light Passes From a Rarer Medium to a Denser Medium

Figure 4 is to be considered for dealing with this particular case of refraction.

In accordance with Step 1 of the algorithm, by choosing a right-handed system of coordinates (not shown in Figure. 4), let us assume that the equation of the concave spherical surface of discontinuity be given by $x^2+ y^2+ z^2= a^2$;or,$\varphi(x,y,z)= x^2+ y^2+ z^2- a^2=0$ and then following Step 2, let us assign the coordinates of the object point A, point of incidence B, point of observation D, and the image point E as(x_1,y_1,z_1), (x_3,y_3,z_3), (x_2,y_2,z_2),and (x_4,y_4,z_4) respectively.

Applying Step 3, it is then necessary to find the following expressions for **i**, **R**, and **n** by considering Figure 4.

$$\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|} = \frac{[(x_3 - x_1)\,\mathbf{I} + (y_3 - y_1)\,\mathbf{J} + (z_3 - z_1)\,\mathbf{K}]}{[U_1]},$$

$$\text{where } U_1 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}\,.$$

$$\mathbf{R} = \frac{\mathbf{ED}}{|\mathbf{ED}|} = \frac{[(x_2 - x_4)\,\mathbf{I} + (y_2 - y_4)\,\mathbf{J} + (z_2 - z_4)\,\mathbf{K}]}{[U_2]},$$

$$\text{where } U_2 = \sqrt{(x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2}\,.$$

$$\mathbf{n} = \frac{[\text{grad}\,\varphi(x,y,z)]_{(x_3,y_3,z_3)}}{|\,[\text{grad}\,\varphi(x,y,z)]_{(x_3,y_3,z_3)}\,|} = \frac{1}{a}\,(x_3\,\mathbf{I} + y_3\,\mathbf{J} + z_3\,\mathbf{K})$$

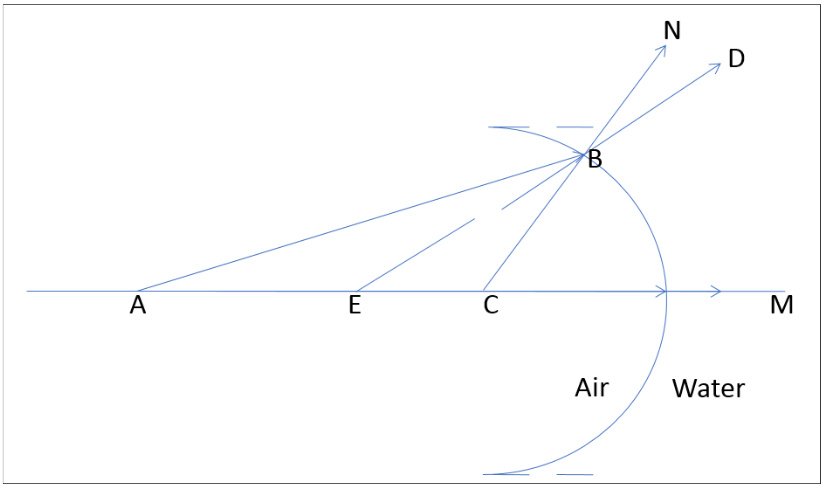

Figure 4: Diagram showing refraction of light at a concave spherical surface of discontinuity as it passes from air to water, in which A is the object point, B is the point of incidence, D is the point of observation, E is the image point, C is the centre of curvature of the convex spherical surface of discontinuity with AM as its principal axis, CN being the positive normal to the surface of discontinuity at the point of incidence

As per Step 4 of the algorithm, it is then necessary to substitute the above expressions for **i**, **R**, and **n** in the generalized vectorial law of refraction (viz. **n** ×**i**=μ(**n** ×**R**)) so as to get the following equation.

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{x_3}{a} & \frac{y_3}{a} & \frac{z_3}{a} \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \mu \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{x_3}{a} & \frac{y_3}{a} & \frac{z_3}{a} \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

or,

$$\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ x_3 & y_3 & z_3 \\ \frac{x_3 - x_1}{U_1} & \frac{y_3 - y_1}{U_1} & \frac{z_3 - z_1}{U_1} \end{vmatrix} = \mu \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ x_3 & y_3 & z_3 \\ \frac{x_2 - x_4}{U_2} & \frac{y_2 - y_4}{U_2} & \frac{z_2 - z_4}{U_2} \end{vmatrix}$$

Following Step 5 of the algorithm, we are then to compare the corresponding coefficients of **I**, **J**, and **K** on both sides of the above relation so as to get the following three sets of equations.

$$\frac{\mu U_1}{U_2} = \frac{\{y_3\,(z_3 - z_1) - z_3\,(y_3 - y_1)\}}{\{y_3\,(z_2 - z_4) - z_3\,(y_2 - y_4)\}} \tag{22}$$

$$\frac{\mu U_1}{U_2} = \frac{\{x_3\,(z_3 - z_1) - z_3\,(x_3 - x_1)\}}{\{x_3\,(z_2 - z_4) - z_3\,(x_2 - x_4)\}} \tag{23}$$

$$\frac{\mu U_1}{U_2} = \frac{\{x_3\,(y_3 - y_1) - y_3\,(x_3 - x_1)\}}{\{x_3\,(y_2 - y_4) - y_3\,(x_2 - x_4)\}} \tag{24}$$

According to Step 6, it is then required to eliminate the common term $\frac{\mu U_1}{U_2}$ from each pair of the above three equations to obtain the following three equations.

$$\frac{\{y_3\,(z_3 - z_1) - z_3\,(y_3 - y_1)\}}{\{y_3\,(z_2 - z_4) - z_3\,(y_2 - y_4)\}} = \frac{\{x_3\,(z_3 - z_1) - z_3\,(x_3 - x_1)\}}{\{x_3\,(z_2 - z_4) - z_3\,(x_2 - x_4)\}} \tag{25}$$

$$\frac{\{x_3\,(z_3 - z_1) - z_3\,(x_3 - x_1)\}}{\{x_3\,(z_2 - z_4) - z_3\,(x_2 - x_4)\}} = \frac{\{x_3\,(y_3 - y_1) - y_3\,(x_3 - x_1)\}}{\{x_3\,(y_2 - y_4) - y_3\,(x_2 - x_4)\}} \tag{26}$$

$$\frac{\{x_3\,(y_3 - y_1) - y_3\,(x_3 - x_1)\}}{\{x_3\,(y_2 - y_4) - y_3\,(x_2 - x_4)\}} = \frac{\{y_3\,(z_3 - z_1) - z_3\,(y_3 - y_1)\}}{\{y_3\,(z_2 - z_4) - z_3\,(y_2 - y_4)\}} \tag{27}$$

In compliance with Step 7, it can now be easily verified that, each of the above three equations (25), (26) and (27) leads to the following determinantal equation for this particular case of refraction, provided that the point of incidence is not the origin of the coordinate system so that x_3, y_3, z_3 are not simultaneously zero.

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \end{vmatrix} = 0 \tag{28}$$

Note 2: Following the same procedure adopted in Case 3 above and making use of the algorithm presented, the readers will be able to arrive at the same determinantal equation (21) or (28) after dealing with the case of reflection at a concave spherical reflecting surface.

Similarly following the same procedure as has been adopted earlier for dealing with Case 4, it can be easily verified that the same determinantal equation (21) or (28) will also result after dealing with each of the following cases of refraction.

(i) Refraction at a concave spherical surface of discontinuity as light passes from a denser medium to a rarer medium having in mind that the generalized vectorial law of refraction to be used for this case is: $\mathbf{n} \times \mathbf{i} = \frac{1}{\mu}(\mathbf{n} \times \mathbf{R})$, where μ is the refractive index of the denser medium with respect to the rarer medium.

(ii) Refraction at a convex spherical surface of discontinuity as light passes from a rarer medium to a denser medium, the generalized vectorial law of refraction to be used for such a case being $\mathbf{n}\times\mathbf{i}=\mu\,(\mathbf{n}\times\mathbf{R})$, where μ is the refractive index of the denser medium with respect to the rarer medium.

(iii) Refraction at a convex spherical surface of discontinuity as light passes from a denser medium to a rarer medium, the generalized vectorial law of refraction to be used for such a case being $\mathbf{n} \times \mathbf{i} = \frac{1}{\mu}(\mathbf{n} \times \mathbf{R})$, where μ is the refractive index of the denser medium with respect to the rarer medium.

Conclusion

Generating determinantal equations (in each of which the coordinates of the object point, point of incidence, point of observation, and the image point are involved) by making use of the generalized vectorial laws of reflection and refraction [1], developed in 2005 is of concern to this paper. Such a problem of generation of relevant determinantal equations in cases of reflection and refraction has never been tried by any one earlier and is therefore novel. A novel algorithm has been presented for the said purpose and that has been subsequently applied to illustrate the procedure of generation of relevant determinantal equations by considering some typical cases of reflection and refraction based on which the readers will be able to arrive at the desired goal while considering all other remaining cases of reflection and refraction.

Unlike the traditional laws of reflection and refraction [5,6], the present scheme proves the efficiency of the generalized vectorial laws of reflection and refraction in respect of generating the relevant determinantal equations.

In addition to novelty and originality of the present scheme, it would be interesting from academic point of view and at the same time it will enrich the optical physics literature thereby enhancing the same as well. Furthermore, the present work is also in support of the fact that the range of applicability of the generalized vectorial laws of reflection and refraction has been increased.

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