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Non-Hermitian Extension of Q Uncertainty Relation

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ABSTRACT

In quantum mechanics it is well known that observables are represented by hermitian matrices (or operators). Uncertainty relations are represented as some kinds of trace inequalities satisfied by two observables and one density matrix (or operator). By releasing the hermitian restriction on observables, we obtained non-hermite uncertainty relations. In this paper we give several non-hermitian extensions of Heisenberg or Schrödinger type q uncertainty relations for generalized skew information under some conditions.

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Introduction

We start from the Heisenberg uncertainty relation:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2$$

for a quantum state (density operator) ρ and two observables (self-adjoint operators) A and B, where $[A,B] = AB - BA$ [1]. The further stronger result was given by Schrödinger in:

$$V_\rho(A)V_\rho(B) - \left|Re\{Cov_\rho(A,B)\}\right|^2 \geq \frac{1}{4}\left|Tr[\rho[A,B]]\right|^2,$$

where the covariance is defined by

$$Cov_\rho(A,B) \equiv Tr\left[\rho\left(A - Tr[\rho A]I\right)\left(B - Tr[\rho B]I\right)\right]$$ [2,3].

The Wigner-Yanase skew information represents a measure for non-commutativity between a quantum state ρ and an observable H [4]. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_\rho(H)\right)^2},$$

with the Wigner-Yanase skew information:

$$I_\rho(H) \equiv \frac{1}{2}Tr\left[(i\left[\rho^{1/2}, H_0\right])^2\right] = Tr\left[\rho H_0^2\right] - Tr\left[\rho^{1/2}H_0\rho^{1/2}H_0\right],$$

$$H_0 \equiv H - Tr[\rho H]I$$

and then he successfully showed a new Heisenberg-type uncertainty relation on $U_\rho(H)$ in:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2. \qquad (1.1)$$

As stated in, the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows [4]. For a mixed state ρ, the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [6,7]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so that we can regard that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity $U_\rho(H)$. After then a one-parameter extension of the inequality (1.1) was given in:

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2,$$

where

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_{\rho,\alpha}(H)\right)^2},$$

with the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$ is defined by

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\left(i\left[\rho^\alpha, H_0\right]\right)\left(i\left[\rho^{1-\alpha}, H_0\right]\right)\right]$$

$$= Tr\left[\rho H_0^2\right] - Tr\left[\rho^\alpha H_0 \rho^{1-\alpha} H_0\right],$$

It is notable that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by Lieb in [8,9]. The further generalization of the Heisenberg-type uncertainty relation on $U_\rho(H)$ has been given in using the generalized Wigner-Yanase-Dyson skew information introduced in [10, 11]. Then it is shown that these skew informations are connected to special choices of quantum Fisher information in [12]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [13]. The Wigner-Yanase skew information and Wigner-Yanase-Dyson skew information are given by the following operator monotone functions

$$f_{WY}(x)=\left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{WYD}(x)=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)},\ \alpha\in(0,1),$$

respectively. In particular the operator monotonicity of the function f_{WYD} was proved in [14]. See also [15]. Recently, Dou and Du proposed the release the restriction on operators which are observables. And they defined the corresponding Wigner-Yanase-Dyson skew information and studied some properties of them in [16,17]. Also they obtained non-hermitian extensions of Heisenberg or Schrödinger uncertainty relations which is a generalization of Luo's theorem. In this paper we give several kinds of non-hermitian extensions of uncertainty relations which correspond to the results given in the case of hermitian observables [18-20].

Operator Monotone Functions

Let $M_n(\mathbb{C})$(resp. Mn,sa ($\mathbb{C}$) be the set of all n x n complex matrices (resp. all n x n self- adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A,B\rangle=Tr(A^*B)$. Let $M_{n,+}(\mathbb{C})$ be the set of all strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of all stricly positive density matrices, that is $M_{n,+,1}(\mathbb{C})=\{\rho\in M_n(\mathbb{C})|Tr\rho=1,\rho\rangle 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho>0$.

A function $f:(0,+\infty)\to\mathbb{R}$ is said to be operator monotone if, for any $n\in\mathbb{N}$ and $A,B\in M_n$ such that $0\le A\le B$, the inequalities $0\le f(A)\le f(B)$ hold. An operator monotone function is said to be symmetric if $f(x)=xf(x^{-1})$ and normalized if $f(1)=1$.

Definition 2.1 $\mathcal{F}_{op}$ is the class of functions $f:(0,+\infty)\to(0,+\infty)$ such that

1. $f(1)=1$,
2. $tf(t^{-1})=f(t)$,
3. f is operator monotone.

Example 2.2 Examples of elements of $\mathcal{F}_{op}$ are given by the following list

$$f_{RLD}(x)=\frac{2x}{x+1},\quad f_{WY}(x)=\left(\frac{\sqrt{x}+1}{2}\right)^2,\quad f_{BKM}(x)=\frac{x-1}{\log x},$$

$$f_{SLD}(x)=\frac{x+1}{2},\quad f_{WYD}(x)=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)},\ \alpha\in(0,1).$$

Remark 2.3 Any $f\in\mathcal{F}_{op}$ satisfies

$$\frac{2x}{x+1}\le f(x)\le\frac{x+1}{2},\quad x>0.$$

For $f\in\mathcal{F}_{op}$ define $f(0)=\lim_{x\to+0}f(x)$. We introduce the sets of all regular and all non-regular functions

$$\mathcal{F}_{op}^r=\{f\in\mathcal{F}_{op}\mid f(0)\ne 0\},\quad \mathcal{F}_{op}^n=\{f\in\mathcal{F}_{op}\mid f(0)=0\}$$

and notice trivially that $\mathcal{F}_{op}=\mathcal{F}_{op}^r\cup\mathcal{F}_{op}^n$

Definition 2.4 Let $g,f\in\mathcal{F}_{op}^r$ satisfy

$$g(x)\ge k\frac{(x-1)^2}{f(x)}\qquad (2.1)$$

for some $k>0$. We define

$$\Delta_g^f(x)=g(x)-k\frac{(x-1)^2}{f(x)}\in\mathcal{F}_{op}$$

Generalized Quasi-Metric Adusted Skew Information and Correlation Measure

In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f\in F_{op}$ by the formula

$$m_f(A,B)=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where $A,B\in M_{n,+}(\mathbb{C})$ Using the notion of matrix means one may define the class of monotone metrics (also called to be quantum Fisher informtions) by the following formula

$$\langle A,B\rangle_{\rho,f}=Tr(A^*\cdot(m_f(L_\rho,R_\rho))^{-1}(B)),$$

where $A,B\in M_n(\mathbb{C}), L_\rho(A)=\rho A, R_\rho(A)=A\rho$. For $q\in\mathbb{R}$ and $A,B\in M(\mathbb{C})$ q-commutator and q-anti-commutator are defined by $[A,B]_q=AB-qBA$ and $\{A,B\}_q=AB+qBA$, respectively.

q-commutator is a generalization of commutator $[A,B]$.

Now we define generalized quasi-metric adjusted q skew information and q correlation measure for non-hermitian matrices $M_n(\mathbb{C})$.

Definition 3.1 For $A,B\in M_n(\mathbb{C})$, $\rho\in M_{n,+,1}(\mathbb{C})$ and q > 0, we define the following quantities, where we put

$$A_0=A-Tr[\rho A]I \text{ and } B_0=B-Tr[\rho B]I.$$

$$Corr_{\rho,q}^{(g,f)}(A,B)=kTr((i[\rho,A_0]_q)^*m_f(L_\rho,qR_\rho)^{-1}i[\rho,B_0]_q),$$

$$I_{\rho,q}^{(g,f)}(A)=Corr_{\rho,q}^{(g,f)}(A,A),$$

$$C_{\rho,q}^f(A,B)=Tr\left(A_0^*m_f(L_\rho,qR_\rho)B_0\right),\quad C_{\rho,q}^f(A)=C_{\rho,q}^f(A,A),$$

$$U_{\rho,q}^{(g,f)}(A)=\sqrt{\left(C_{\rho,q}^g(A)+C_{\rho,q}^{\Delta_g^f}(A)\right)\left(C_{\rho,q}^g(A)-C_{\rho,q}^{\Delta_g^f}(A)\right)},$$

The quantity $I_{\rho,q}^{(g,f)}(A)$ and $Corr_{\rho,q}^{(g,f)}(A,B)$ are said generalized quasi-metric adjusted q skew information and generalized quasi-metric adjusted q correlation measure, respectively.

Then we have the following proposition.

Proposition 3.2 For $A,B\in M_n(\mathbb{C})$, $\rho\in M_{n,+,1}(\mathbb{C})$ and $q>0$, we have the following relations:

1. $I_{\rho,q}^{(g,f)}(A)=C_{\rho,q}^g(A)-C_{\rho,q}^{\Delta_g^f}(A)$
2. $J_{\rho,q}^{(g,f)}(A)=C_{\rho,q}^g(A)+C_{\rho,q}^{\Delta_g^f}(A)$
3. $U_{\rho,q}^{(g,f)}(A)=\sqrt{I_{\rho,q}^{(g,f)}(A)J_{\rho,q}^{(g,f)}(A)}$

Theorem 3.3 For $f\in\mathcal{F}_{op}^r$, it holds

$$I_{\rho,q}^{(g,f)}(A) I_{\rho,q}^{(g,f)}(B) \geq |Corr_{\rho,q}^{(g,f)}(A,B)|^2,$$

where $A,B \in M_n(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$ and $q>0$.

Proof of Theorem 3.3. Since

$$Corr_{\rho,q}^{(g,f)}(A,B) = kTr((i[\rho,A_0]_q)^* m_f(L_\rho,qR_\rho)^{-1} i[\rho,B_0]_q)$$

$$= kTr((i(L_\rho - qR_\rho)A_0)^* m_f(L_\rho,qR_\rho)^{-1} i(L_\rho - qR_\rho)B_0)$$

$$= Tr(A_0^* m_g(L_\rho,qR_\rho)B_0) - Tr(A_0^* m_{\Delta_g^f}(L_\rho,qR_\rho)B_0),$$

it is easy to show that $Corr_{\rho,q}^{(g,f)}(A,B)$ is an inner product in $M_n(\mathbb{C})$. Then we can get the result by using Schwarz inequality.

Theorem 3.4 For $f \in F_{op}^r$, if

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \qquad (3.1)$$

for some $\ell > 0$, then it holds

$$U_{\rho,q}^{(g,f)}(A) U_{\rho,q}^{(g,f)}(B) \geq k\ell |Tr(\rho[A_0,B_0]_q)|^2, \qquad (3.2)$$

where $A,B \in M_n(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$ and $q>0$.

In order to prove Theorem 3.4, we need the following lemmas

Lemma 3.5 If (2.1) and (3.1) are satisfied, then we have the following inequality:

$$m_g(x,qy)^2 - m_{\Delta_g^f}(x,qy)^2 \geq k\ell(x-qy)^2.$$

Proof of Lemma 3.5: By (2.1) and (3.1), we have

$$m_{\Delta_g^f}(x,qy) = m_g(x,qy) - k\frac{(x-qy)^2}{m_f(x,qy)}. \qquad (3.3)$$

$$m_g(x,qy) + m_{\Delta_g^f}(x,qy) \geq \ell m_f(x,qy), \qquad (3.4)$$

Therefore by (3.3), (3.4)

$$m_g(x,qy)^2 - m_{\Delta_g^f}(x,qy)^2$$

$$= \{m_g(x,qy) - m_{\Delta_g^f}(x,qy)\}\{m_g(x,qy) + m_{\Delta_g^f}(x,qy)\}$$

$$\geq k\frac{(x-qy)^2}{m_f(x,qy)} \ell m_f(x,qy)$$

$$= k\ell(x-qy)^2.$$

Lemma 3.6 Let $\{|\phi_1\rangle, |\phi_2\rangle, \cdots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$. We put $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle, b_{ij} = \langle\phi_i|B_0|\phi_j\rangle$, where $A_0 \equiv A - Tr(\rho A)I$ and $B_0 \equiv B - Tr(\rho B)I$ for $A,B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Then we have

$$I_{\rho,q}^{(g,f)}(A) = \sum_{i,j} m_g(\lambda_i,q\lambda_j)|a_{ij}|^2 - \sum_{i,j} m_{\Delta_g^f}(\lambda_i,q\lambda_j)|a_{ij}|^2 .$$

$$J_{\rho,q}^{(g,f)}(A) = \sum_{i,j} m_g(\lambda_i,q\lambda_j)|a_{ij}|^2 + \sum_{i,j} m_{\Delta_g^f}(\lambda_i,q\lambda_j)|a_{ij}|^2 .$$

$$U_{\rho,q}^{(g,f)}(A)^2 = \left(\sum_{i,j} m_g(\lambda_i,q\lambda_j)|a_{ij}|^2\right)^2 - \left(\sum_{i,j} m_{\Delta_g^f}(\lambda_i,q\lambda_j)|a_{ij}|^2\right)^2$$

and

$$Corr_{\rho,q}^{(g,f)}(A,B) = \sum_{i,j} m_g(\lambda_i,q\lambda_j)\overline{a_{ij}}b_{ij} - \sum_{i,j} m_{\Delta_g^f}(\lambda_i,q\lambda_j)\overline{a_{ij}}b_{ij}.$$

We are now in a position to prove Theorem 3.4.

Proof of Theorem 3.4: At first we prove (3.3). Since

$$Tr(\rho[A_0,B_0]_q) = \sum_{i,j}(\lambda_i - q\lambda_j)a_{ij}b_{ji},$$

$$|Tr(\rho[A_0,B_0]_q)| \leq \sum_{i,j}|\lambda_i - q\lambda_j||a_{ij}||b_{ji}|.$$

Then by Lemma 3.5, we have

$$k\ell |Tr(\rho[A_0,B_0]_q)|^2$$

$$\leq \left\{\sum_{i,j}\sqrt{k\ell}|\lambda_i - q\lambda_j||a_{ij}||b_{ji}|\right\}^2$$

$$\leq \left\{\sum_{i,j}\left(m_g(\lambda_i,q\lambda_j)^2 - m_{\Delta_g^f}(\lambda_i,q\lambda_j)^2\right)^{1/2}|a_{ij}||b_{ji}|\right\}^2$$

$$\leq \left\{\sum_{i,j}\left(m_g(\lambda_i,q\lambda_j)^2 - m_{\Delta_g^f}(\lambda_i,q\lambda_j)^2\right)|a_{ij}|^2\right\}$$

$$\left\{\sum_{i,j}\left(m_g(\lambda_i,q\lambda_j)^2 + m_{\Delta_g^f}(\lambda_i,q\lambda_j)^2\right)|b_{ji}|^2\right\}$$

$$= I_{\rho,q}^{(g,f)}(A) J_{\rho,q}^{(g,f)}(B).$$

By the similar way, we also have

$$I_{\rho,q}^{(g,f)}(B) J_{\rho,q}^{(g,f)}|(A) \geq k\ell|Tr(\rho[A_0,B_0]_q)|^2 .$$

Hence we have the desired inequality (3.2).

Examples

Example 4.1

When

$$g(x) = \frac{x+1}{2},\ f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},\ k = \frac{f(0)}{2},\ \ell = 2,$$

and $A,B \in M_n(\mathbb{C})$, we give the following:

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} = \frac{1}{2}(x^\alpha + x^{1-\alpha}) \geq 0.$$

$$g(x) + \Delta_g^f(x) - \ell f(x)$$

$$= \frac{1}{2\left(x^{\alpha}-1\right)\left(x^{1-\alpha}-1\right)}\{\left(x^{2\alpha}-1\right)\left(x^{2(1-\alpha)}-1\right)-4\alpha\left(1-\alpha\right)(x-1)^2\} \geq 0.$$

Then

$$I^{(f\ g)}\left(A\right)$$

$$= \frac{1}{2}\left\{Tr\left(\rho A_0 A_0^*\right)+qTr\left(\rho A_0^* A_0\right)-q^{1-\alpha}Tr\left(\rho^{\alpha}A_0\rho^{1-\alpha}A_0^*\right)-q^{\alpha}Tr\left(\rho^{\alpha}A_0^*\rho^{1-\alpha}A_0\right)\right\}.$$

In particular for $\alpha = 1/2$,

$$I_{\rho,q}^{(f,g)}\left(A\right)=\frac{1}{2}\left\{Tr\left(\rho A_0 A_0^*\right)+qTr\left(\rho A_0^* A_0\right)\right\}-q^{1/2}Tr\left(\rho^{1/2}A_0\rho^{1/2}A_0^*\right).$$

Example 4.2
When

$$g\left(x\right)=\left(\frac{\sqrt{x}+1}{2}\right)^2,\ f\left(x\right)=\alpha\left(1-\alpha\right)\frac{(x-1)^2}{\left(x^{\alpha}-1\right)\left(x^{1-\alpha}-1\right)}.$$

and $A,B\in M_n(\mathbb{C})$, we assume k = f (0)/8 and $\ell = 3/2$, then we have the following.

$$\Delta_g^f\left(x\right)=g\left(x\right)-k\frac{(x-1)^2}{f\left(x\right)}=\left(\frac{1+\sqrt{x}}{2}\right)^2-\frac{1}{8}\left(x^{\alpha}-1\right)\left(x^{1-\alpha}-1\right)$$

$$=\frac{1}{8}\{(1+\sqrt{x})^2+(x^{\alpha/2}+x^{(1-\alpha)/2})^2\}\geq 0.$$

$$g\left(x\right)+\Delta_g^f\left(x\right)-\ell f\left(x\right)$$

$$=2g\left(x\right)-\frac{1}{8}\left(x^{\alpha}-1\right)\left(x^{1-\alpha}-1\right)-\frac{3}{2}f\left(x\right)$$

$$\geq\frac{1}{2}g\left(x\right)-\frac{1}{8}\left(x^{\alpha}-1\right)\left(x^{1-\alpha}-1\right)$$

$$=\frac{1}{8}(x^{\alpha/2}+x^{(1-\alpha)/2})^2\geq 0.$$

Example 4.3
When

$$g\left(x\right)=\left(\frac{x^{\gamma}+1}{2}\right)^{1/\gamma}\ \left(\frac{3}{4}\leq\gamma\leq 1\right),\ f\left(x\right)=\left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$k=\frac{f\left(0\right)}{4},\ell=2,$$

and $A,B\in M_n(\mathbb{C})$, we give the following: Let

$$F\left(x,r\right)=\left(\frac{1+x^r}{2}\right)^{1/r}.$$

Since $F(x,\cdot)$ is concave on [1/ 2,1] [20].

$$F\left(t,\frac{3}{4}\right)\geq\frac{1}{2}F\left(t,1\right)+\frac{1}{2}F\left(t,\frac{1}{2}\right).$$

Then

$$2F\left(x,r\right)\geq 2F\left(x,\frac{3}{4}\right)\geq F\left(x,1\right)+F\left(x,\frac{1}{2}\right),$$

That is

$$2\left(\frac{1+x^r}{2}\right)^{1/r}-\left(\frac{\sqrt{x}-1}{2}\right)^2>2\left(\frac{\sqrt{x}+1}{2}\right)^2.$$

Then since

$$\Delta_g^f\left(x\right)=\left(\frac{1+x^r}{2}\right)^{1/r}-\left(\frac{\sqrt{x}-1}{2}\right)^2,$$

we have

$$g\left(x\right)+\Delta_g^f\left(x\right)\geq 2f\left(x\right).$$

Example 4.4
When

$$g\left(x\right)=\left(\frac{1+x^r}{2}\right)^{1/r},\ \left(\frac{5}{8}\leq r\leq 1\right)$$

$$f\left(x\right)=\left(\frac{1+\sqrt{x}}{2}\right)^2,\quad k=\frac{f\left(0\right)}{8}=\frac{1}{32},\quad \ell=2,$$

we give the following. Since $F(x,\cdot)$ is concave on [1/2,3/4] [20].

$$F\left(x,\frac{5}{8}\right)\geq\frac{1}{2}F\left(x,\frac{1}{2}\right)+\frac{1}{2}F\left(x,\frac{3}{4}\right).$$

Then

$$2F\left(x,r\right)\geq 2F\left(x,\frac{5}{8}\right)\geq F\left(x,\frac{1}{2}\right)+F\left(x,\frac{3}{4}\right)$$

$$\geq F\left(x,\frac{1}{2}\right)+\frac{1}{2}\left\{\frac{x+1}{2}+\left(\frac{\sqrt{x}+1}{2}\right)^2\right\}$$

$$=\frac{3}{2}\left(\frac{\sqrt{x}+1}{2}\right)^2+\frac{1}{2}\frac{x+1}{2}$$

$$=\frac{3}{2}\left(\frac{\sqrt{x}+1}{2}\right)^2+\frac{1}{2}\left\{\left(\frac{\sqrt{x}-1}{2}\right)^2+\left(\frac{\sqrt{x}+1}{2}\right)^2\right\}$$

$$=2\left(\frac{\sqrt{x}+1}{2}\right)^2+\frac{1}{2}\left(\frac{\sqrt{x}-1}{2}\right)^2$$

Thus we have

$$g\left(x\right)+\Delta_g^f\left(x\right)\geq 2f\left(x\right).$$

Application
Under the assumption in Example 4.1, we can give an application of Theorem 3.3. In [3] some non-hermitian uncertainty relations were obtained. One of their q-versions is given as a corollary of Theorem 3.4 [21-28].

Corollary 5.1 For $A, B \in M_n(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$ and q > 0,

$$U_{\rho,q}^{(g,f)}(A)U_{\rho,q}^{(g,f)}(B) \geq \alpha(1-\alpha)|Tr(\rho[A_0,B_0]_q^0)|^2,$$

where $[A_0,B_0]_q^0 = \frac{1}{2}\{[A_0,B_0]_q \pm [B_0^*,A_0^*]_q\}$

Proof of Corollary 5.1. When

$[A_0,B_0]_q^0 = \frac{1}{2}\{Tr([A_0,B_0]_q + [B_0^*,A_0^*]_q\}$, we have by Theorem 3.4

$$\begin{aligned} U_{\rho,q}^{(g,f)}(A)U_{\rho,q}^{(g,f)}(B) &\geq \alpha(1-\alpha)|Tr(\rho[A_0,B_0]_q)|^2 \\ &\geq \alpha(1-\alpha)|Re\{Tr(\rho[A_0,B_0]_q)\}|^2 \\ &= \alpha(1-\alpha)|Tr(\rho[A_0,B_0]_q^0)|^2 . \end{aligned}$$

The last equality holds for the following reason. Since

$$\begin{aligned} Tr(\rho[B_0^*,A_0^*]_q) &= Tr(\rho B_0^* A_0^*) - qTr(\rho A_0^* B_0^*) \\ &= \overline{Tr(A_0 B_0 \rho)} - q\overline{Tr(B_0 A_0 \rho)} \\ &= \overline{Tr(\rho A_0 B_0)} - q\overline{Tr(\rho B_0 A_0)} \\ &= \overline{Tr(\rho(A_0 B_0 - qB_0 A_0))} \\ &= \overline{Tr(\rho[A_0,B_0]_q)}, \end{aligned}$$

we have

$$\begin{aligned} Tr(\rho[A_0,B_0]_q^0) &= \frac{1}{2}Tr(\rho[A_0,B_0]_q) + \frac{1}{2}Tr(\rho[B_0^*,A_0^*]_q) \\ &= \frac{1}{2}Tr(\rho[A_0,B_0]_q) + \frac{1}{2}\overline{Tr(\rho[A_0,B_0]_q)} \\ &= Re\{Tr(\rho[A_0,B_0]_q)\}. \end{aligned}$$

On the other hand when $[A_0,B_0]_q^0 = \frac{1}{2}\{Tr([A_0,B_0]_q - [B_0^*,A_0^*]_q)\}$,

we obtain the desired inequality (5.1) by the similar way.

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