Non-Hermitian Extension of Q-Uncertainty Relation

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ABSTRACT
In quantum mechanics it is well known that observables are represented by hermitian matrices (or operators). Uncertainty relations are represented as some kind of trace inequalities satisfied by two observables and one density matrix (or operator). By releasing the hermitian restriction on observables, we obtained non-hermitian uncertainty relations. In this paper we give several non-hermitian extensions of Heisenberg or Schrödinger type q uncertainty relations for generalized skew information under some conditions.

Keywords: Uncertainty Relation, Q-Commutator

Introduction
We start from the Heisenberg uncertainty relation:

\[ V_\rho (A)V_\rho (B) \geq \frac{1}{4} |\text{Tr}[\rho [A,B]]|^2 \]

for a quantum state (density operator) \( \rho \) and two observables (self-adjoint operators) \( A \) and \( B \), where \( [A,B]=AB-BA \). The further stronger result was given by Schrödinger in:

\[ V_\rho (A)V_\rho (B) - \text{Re}\{\text{Cov}_\rho (A,B)\}^2 \geq \frac{1}{4} |\text{Tr}[\rho [A,B]]|^2 ,\]

where the covariance is defined by

\[ \text{Cov}_\rho (A,B) = \text{Tr}[\rho (A - \text{Tr}[\rho A]) (B - \text{Tr}[\rho B])]. \]

The Wigner-Yanase skew information represents a measure for non-commutativity between a quantum state \( \rho \) and an observable \( H \) [4]. Luo introduced the quantity \( U_\rho (H) \) representing a quantum uncertainty excluding the classical mixture:

\[ U_\rho (H) = \sqrt{\rho (H)^2 -(\text{Tr}[\rho H])^2} \]

with the Wigner-Yanase skew information:

\[ I_\rho (H) = \frac{1}{2} \text{Tr}\left( i[H,\rho^{1/2}H_0^{1/2}] \right) = \text{Tr}\left( \rho H_0^{1/2}H\rho^{1/2}H_0 \right) - \text{Tr}\left( \rho^{1/2}H_0^{1/2}H_0\right). \]

It is notable that the convexity of \( I_\rho (H) \) with respect to \( \rho \) was successfully proven by Lieb in [8,9]. The further generalization of the Heisenberg-type uncertainty relation on \( U_\rho (H) \) has been given in using the generalized Wigner-Yanase-Dyson skew information introduced in [10,11]. Then it is shown that these skew informations are connected to special choices of quantum Fisher information in [12]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions \( \mathcal{F}_\text{op} \) which were justified in [13].

As stated in, the physical meaning of the quantity \( U_\rho (H) \) can be interpreted as follows [4]. For a mixed state \( \rho \), the variance \( V_\rho (H) \) has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information \( I_\rho (H) \) represents a kind of quantum uncertainty [6,7]. Thus, the difference \( V_\rho (H) - I_\rho (H) \) has a classical mixture so that we can regard that the quantity \( U_\rho (H) \) has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity \( U_\rho (H) \). After then a one-parameter extension of the inequality (1.1) was given in:

\[ U_{\rho,\alpha} (A)U_{\rho,\alpha} (B) \geq \alpha (1-\alpha) |\text{Tr}[\rho [A,B]]|^2 ,\]

where

\[ U_{\rho,\alpha} (H) = \sqrt{V_{\rho,\alpha} (H)^2 -(\text{Tr}[\rho H])^2} \]

with the Wigner-Yanase-Dyson skew information \( I_{\rho,\alpha} (H) \) is defined by

\[ I_{\rho,\alpha} (H) = \frac{1}{2} \text{Tr}\left( i[H,\rho^{1/2}H_0^{1/2}] \right) \text{Tr}\left( \rho^{1/2}H_0^{1/2}H_0\right) \]

and then he successfully showed a new Heisenberg-type uncertainty relation on \( U_{\rho,\alpha} (H) \) in:

\[ U_{\rho,\alpha} (A)U_{\rho,\alpha} (B) \geq \frac{1}{4} |\text{Tr}[\rho [A,B]]|^2 . \]
Operator Monotone Functions
Let $M_n(M)$ (resp. $M_n,sa(M)$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $(A,B) = \text{Tr}(A^*B)$. Let $M_{s,n}(M)$ be the set of all strictly positive elements of $M_n(M)$ and $M_{s,1}(M)$ be the set of all strictly positive density matrices, that is $M_{s,n}(M) = \{ \rho\in M_n(M) | \rho > 0 \}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho > 0$.

A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is said to be operator monotone if, for any $n \in \mathbb{N}$ and $A,B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said to be symmetric if $f(x) = x f(x^{-1})$ and normalized if $f(1) = 1$.

**Definition 2.1** $\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

1. $f(1) = 1$,
2. $f(t^{-1}) = f(t)$,
3. $f$ is operator monotone.

**Example 2.2** Examples of elements of $\mathcal{F}_{op}$ are given by the following list

- $f_{\text{exp}} (x) = \frac{2x}{x + 1}$, $f_{\text{exp}} (x) = \frac{x^2}{2}$, $f_{\text{exp}} (x) = \frac{x - 1}{\log x}$, $f_{\text{exp}} (x) = \frac{x - 1}{\log x}$, $f_{\text{exp}} (x) = \frac{x - 1}{\log x}$, $f_{\text{exp}} (x) = \frac{x - 1}{\log x}$, $f_{\text{exp}} (x) = \frac{x - 1}{\log x}$, $f_{\text{exp}} (x) = \frac{x - 1}{\log x}$.

**Remark 2.3** Any $f \in \mathcal{F}_{op}$ satisfies

\[
2 \leq f(x) \leq \frac{x + 1}{2}, \quad x > 0.
\]

For $f \in \mathcal{F}_{op}$ define $f(0) = \lim_{x \to 0} f(x)$. We introduce the sets of all regular and all non-regular functions

- $\mathcal{F}_{op}^r = \{ f \in \mathcal{F}_{op} | f(0) \neq 0 \}$,
- $\mathcal{F}_{op}^n = \{ f \in \mathcal{F}_{op} | f(0) = 0 \}$

and notice trivially that $\mathcal{F}_{op} = \mathcal{F}_{op}^r \cup \mathcal{F}_{op}^n$.

**Definition 2.4** Let $g, f \in \mathcal{F}_{op}^r$ satisfy

\[
g(x) \geq k \frac{(x - 1)^2}{f(x)}
\]

for some $k > 0$. We define

\[
\Delta_f^g(x) = g(x) - k \frac{(x - 1)^2}{f(x)} \in \mathcal{F}_{op}
\]

**Generalized Quasi-Metric Adjusted Skewed Information and Correlation Measure**

In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathcal{F}_{op}$ by the formula

\[
m_f(A,B) = A^{1/2} f\left(A^{1/2} B A^{1/2}\right) A^{1/2},
\]

where $A,B \in M_{s,n}(M)$ Using the notion of matrix means one may define the class of monotone metrics (also called to be quantum Fisher informations) by the following formula

\[
\langle A,B\rangle_f = \text{Tr}\left(A^{1/2} B A^{1/2}\right) f\left(A^{1/2} B A^{1/2}\right).
\]

Now we define generalized quasi-metric adjusted $q$ skew information and $q$ correlation measure for non-hermitian matrices $M_{s,1}(M)$.

**Definition 3.1** For $A,B \in M_{s,1}(M), \rho \in M_{s,1,1}(M)$ and $q > 0$, we define the following quantities, where we put

\[
A_\rho = A - \text{Tr}(\rho A) I \quad \text{and} \quad B_\rho = B - \text{Tr}(\rho B) I.
\]

\[
\text{Cor}_{\rho}^{(s,f)}(A,B) = k\text{Tr}((\rho[A,B])^* m_f(L_{\rho},qR_{\rho}))^1 f[\rho B]\}
\]

\[
I^{(s,f)}_{\rho}(A) = \text{Cor}_{\rho}^{(s,f)}(A,A),
\]

\[
C_{\rho}^{(s,f)}(A,B) = \text{Tr}\left(A_\rho m_f(L_{\rho},qR_{\rho})B_\rho\right),
\]

\[
U^{(s,f)}_{\rho}(A) = \sqrt{C_{\rho}^{(s,f)}(A) + C_{\rho}^{(s,f)}(A)} C_{\rho}^{(s,f)}(A) - C_{\rho}^{(s,f)}(A).
\]

The quantity $I^{(s,f)}_{\rho}(A)$ and $C_{\rho}^{(s,f)}(A,B)$ are said generalized quasi-metric adjusted $q$ skew information and generalized quasi-metric adjusted $q$ correlation measure, respectively.

Then we have the following proposition.

**Proposition 3.2** For $A,B \in M_{s,1}(M), \rho \in M_{s,1,1}(M)$ and $q > 0$, we have the following relations:

1. $I^{(s,f)}_{\rho}(A) = C_{\rho}^{(s,f)}(A) - C_{\rho}^{(s,f)}(A)$
2. $I^{(s,f)}_{\rho}(A) = C_{\rho}^{(s,f)}(A) + C_{\rho}^{(s,f)}(A)$
3. $U^{(s,f)}_{\rho}(A) = \sqrt{I^{(s,f)}_{\rho}(A) I^{(s,f)}_{\rho}(A)}$
By (2.1) and (3.1), we have the following inequality:

**Lemma 3.5** If (2.1) and (3.1) are satisfied, then we have the following inequality:

$$\sum_{i,j} m_g(\lambda_i, q\lambda_j)|a_{ij}|^2 \leq \sum_{i,j} m_{\Delta'_g}(\lambda_i, q\lambda_j)|a_{ij}|^2$$

**Proof of Theorem 3.4.** Since

$$\text{Corr}_{\rho q}^{(g,f)}(A,B) = \text{Tr}(\{|f_{\rho q}(\Psi)|^2 \}^2) - k \text{Tr}(\rho[A_{\|B}]_q)$$

we can get the result by using Schwarz inequality.

Therefore by (3.3), (3.4)

$$m_g(x,qy^2) - m_{\Delta'_g}(x,qy^2) \geq k\ell(x-qy^2)^2.$$
Then

\[ I^{(f,g)}(A) \]

\[ = \frac{1}{2} \left[ \text{Tr} \left( \rho A_\alpha A_\beta \right) + q \text{Tr} \left( \rho A_\alpha A_\beta \right) - q^{1/2} \text{Tr} \left( \rho^{1/2} A_\alpha A_\beta \right) - q^{1/2} \text{Tr} \left( \rho^{1/2} A_\alpha A_\beta \right) \right]. \]

In particular for \( \alpha = 1/2 \),

\[ I^{(f,g)}_{1/2}(A) = \frac{1}{2} \left[ \text{Tr} \left( \rho A_\alpha A_\beta \right) + q \text{Tr} \left( \rho A_\alpha A_\beta \right) - q^{1/2} \text{Tr} \left( \rho^{1/2} A_\alpha A_\beta \right) - q^{1/2} \text{Tr} \left( \rho^{1/2} A_\alpha A_\beta \right) \right]. \]

Example 4.2

When

\[ g(x) = \left( \frac{x^{1/2} + 1}{2} \right)^2, \quad f(x) = (1 - \alpha) \frac{(x-1)^2}{\left( x^{1/2} - 1 \right) \left( x^{1/2} - 1 \right)} . \]

and \( A, B \in M_n(C) \), we assume \( k = f(0)/8 \) and \( \ell = 3/2 \), then we have the following.

\[ \Delta_{g}^\prime (x) = g(x) - k \frac{(x-1)^2}{f(x)} = \left( \frac{1 + x^{1/2}}{2} \right)^2 - \frac{1}{8} \left( x^{1/2} - 1 \right)^2 \]

\[ = \frac{1}{8} \left\{ (1 + x^{1/2})^2 + (x^{1/2} + x^{1/2})^2 \right\} \geq 0. \]

\[ g(x) + \Delta_{g}^\prime (x) \geq f(x) \]

\[ = 2g(x) - \frac{1}{8} \left( x^{1/2} - 1 \right)^2 \left( x^{1/2} - 1 \right) - \frac{3}{2} f(x) \]

\[ \geq \frac{1}{2} g(x) - \frac{1}{8} \left( x^{1/2} - 1 \right)^2 \left( x^{1/2} - 1 \right) \]

\[ = \frac{1}{8} \left( x^{1/2} + x^{1/2} \right)^2 \geq 0. \]

Example 4.3

When

\[ g(x) = \left( \frac{x^{1/2} + 1}{2} \right)^{\gamma}, \quad f(x) = \left( \frac{x^{1/2} + 1}{2} \right)^2, \]

\[ k = \frac{f(0)}{4}, \ell = 2, \]

and \( A, B \in M_n(C) \), we give the following: Let

\[ F(x, r) = \left( \frac{1 + x^{1/2}}{2} \right)^{\gamma}. \]

Since \( F(x,.) \) is concave on \([1/2, 1]\) [20].

\[ F\left( r, \frac{3}{4} \right) \geq \frac{1}{2} F\left( r, 1 \right) + \frac{1}{2} F\left( r, \frac{1}{2} \right). \]

Then

\[ 2F(x, r) \geq 2F\left( x, \frac{3}{4} \right) \geq F(x, 1) + F\left( x, \frac{1}{2} \right). \]

That is

\[ 2 \left( \frac{1 + x^{1/2}}{2} \right)^{\gamma} - \left( \frac{\sqrt{x} - 1}{2} \right)^2 \geq 2 \left( \frac{\sqrt{x} + 1}{2} \right)^2. \]

Then since

\[ \Delta_{g}^\prime (x) = \left( \frac{1 + x^{1/2}}{2} \right)^{\gamma} - \left( \frac{\sqrt{x} - 1}{2} \right)^2, \]

we have

\[ g(x) + \Delta_{g}^\prime (x) \geq 2f(x). \]

Example 4.4

When

\[ g(x) = \left( \frac{1 + x^{1/2}}{2} \right)^{\gamma}, \quad \left( \frac{5}{8} \leq r \leq 1 \right) \]

\[ f(x) = \left( \frac{1 + x^{1/2}}{2} \right)^2, \quad k = \frac{f(0)}{8} = \frac{1}{32}, \ell = 2, \]

we give the following. Since \( F(x,.) \) is concave on \([1/2, 3/4]\) [20].

\[ F\left( x, \frac{5}{8} \right) \geq \frac{1}{2} F\left( x, \frac{1}{2} \right) + \frac{1}{2} F\left( x, \frac{3}{4} \right). \]

Then

\[ 2F(x, r) \geq 2F\left( x, \frac{5}{8} \right) \geq F\left( x, \frac{1}{2} \right) + F\left( x, \frac{3}{4} \right) \geq F\left( x, \frac{1}{2} \right) + \frac{1}{2} \left( \frac{\sqrt{x} + 1}{2} \right)^2 \]

\[ \geq 3 \left( \frac{\sqrt{x} + 1}{2} \right)^2 + 1 \frac{x+1}{2} \]

\[ = 3 \left( \frac{\sqrt{x} + 1}{2} \right)^2 + \frac{1}{2} \left( \frac{\sqrt{x} + 1}{2} \right)^2 \]

\[ = 2 \left( \frac{\sqrt{x} + 1}{2} \right)^2 + \frac{1}{2} \left( \frac{\sqrt{x} - 1}{2} \right)^2 \]

Thus we have

\[ g(x) + \Delta_{g}^\prime (x) \geq 2f(x). \]

Application

Under the assumption in Example 4.1, we can give an application of Theorem 3.3. In [3] some non-hermitian uncertainty relations were obtained. One of their q-versions is given as a corollary of Theorem 3.4 [21-28].
Corollary 5.1 For $A, B \in M_n(C)$, $\rho \in M_{n, \mathbb{C}}$ and $q > 0,$

$$U_{\rho, q}^{(g.r)}(A)U_{\rho, q}^{(g.r)}(B) \geq \alpha(1 - \alpha)|Tr(\rho[A_n, B_n])|^2,$$

where $[A_n, B_n] = \frac{1}{2}[\{A_n, B_n\} + \{B_n, A_n\}].$

**Proof of Corollary 5.1.** When

$$[A_n, B_n] = \frac{1}{2}(Tr([A_n, B_n] + [B_n, A_n]),$$

we have by Theorem 3.4

\begin{align*}
U_{\rho, q}^{(g.r)}(A)U_{\rho, q}^{(g.r)}(B) & \geq \alpha(1 - \alpha)|Tr(\rho[A_n, B_n])|^2 \\
& \geq \alpha(1 - \alpha)|Re(Tr(\rho[A_n, B_n])|^2 \\
& = \alpha(1 - \alpha)|Tr(\rho[A_n, B_n])^2|.
\end{align*}

The last equality holds for the following reason. Since

$$Tr(\rho[B_n^*, A_n^*]) = Tr(\rho B_n^* A_n) - qTr(\rho A_n^* B_n),$$

we have

$$Tr(\rho[B_n^*, A_n^*]) = \frac{1}{2}Tr(\rho[B_n^*, A_n^*]) + \frac{1}{2}Tr(\rho[A_n^*, B_n^*]).$$

On the other hand when $[A_n, B_n] = \frac{1}{2}[\{A_n, B_n\}]$, we obtain the desired inequality (5.1) by the similar way.

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**References**