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Integrals Involving Jacobi Theta Functions and Applications in Quantum Systems

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ABSTRACT

We study four expressions involving the integrals of Jacobi's theta functions. From Poisson's summation formula, we write the integrals of the functions θ_i, (i = 1,2,3,4) in terms of modified Bessel functions of the second kind. For the integrals of θ_1, θ_2 and θ_3, we get expressions with real arguments, but for the integral of θ_4, we find an expression with imaginary argument. In addition, we apply our results to the description of two kinds of interacting quantum systems: boson gas and fermion gas both under a thermal bath and an external magnetic field.

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Introduction

Almost 200 years ago, Jacobi investigated some functions known today as theta functions: $\theta_i(z; q)$ with (z,q) ∈ C, |q| < 1 (i = 1,2,3,4) and demonstrated their main properties [1, 2]. In the last years, Jacobi theta functions have been used in the context of heat conduction theory, for problems related to orthogonalization and interpolation, in probability laws, in a class of hypergeometric integrals, in the complete elliptic integral of the first kind K(k) and in quantum field theory (QFT) at finite temperature with boundary conditions [3-10].

Through Schwinger's proper-time representation for the Feynman propagator, it is possible to write the integrals on the proper-time s in terms of integrals on the Jacobi theta functions. However, these integrals are only tractable from a numerical point of view (up to our knowledge). Indeed, the integrals involving the Jacobi theta functions that appear in QFT acquire infinite values for the argument s → 0, thus requiring a regularization process of these integrals, for a later solution numeric [11-13].

The integrals of $\theta_i\,[f(s); g(s)]$ with s in the range [0,+∞] which we will compute in this note, play an important role in effective models of quantum chromodynamics (QCD) (for a review these effective models of QCD, see for example the Refs. [14-16]). This protagonism arises when we apply the generalized Matsubara formalism to the model, which allows us to introduce the spatial boundaries, temperature, chemical potential, and magnetic effects in the quantum system (the reader can find a review of generalized Matsubara formalism in) [17, 18].

In this contribution, we produce a closed form for these integrals, in terms of well-behaved Bessel functions of the second type. This will be done considering the physical interest of applying our results in Quantum Mechanics (QM) with and without magnetic effects. For this reason, we will define the functions $f(s)$ and $g(s)$ as ($\equiv abs$) and [$\equiv \exp(-a^2 s)$], respectively. This choice allows us to include in the systems, thermodynamic variables such as temperature and chemical potential by the formalism of imaginary-time, for boson fields (even frequencies) and for fermion fields (odd frequencies).

This approach establishes the QFT in the topology $\Gamma^1{}_D = S^1\times\mathbf{R}^{D-1}$, where the S^1 represents a compact in the imaginary-time direction τ and there is no restriction on the remaining spatial coordinates $(D-1)$. From the boundary conditions for the bosonic or fermionic fields at coordinate τ, we can demonstrate that S^1 defines a circle with radius $\beta/2\pi$ where β is the inverse of system temperature, that is, $\beta = 1/k_B T$. Therefore, in the imaginary-time formalism, $\tau \in [0,\beta]$.

The paper is structured as follows: In Section II, we obtain explicit expressions for the integrals of all Jacobi's theta functions in terms of exponentially decaying functions $K_\nu(x)$. In Section III, we apply the results in two different and self-interacting quantum systems:

the first one formed by a gas of bosons and the other one composed by a gas of fermions. We shall define the Hamiltonian density for the cited systems and use some QFT results, in particular, the expressions for the Feynman propagator when the quantum field is under an external magnetic field. In this scenario, the so-called Landau levels arise (quantized energy levels in the plane orthogonal to the applied external magnetic field). Also, we explore the findings in the context of phase transition and broken/restoration of symmetry. We conclude this work and make some considerations about it, in Section IV. At physical applications, we considered a four-dimensional Euclidean space and the natural unit system such that $c = \hbar = k_B = 1$.

Integrals of Theta Functions

Let us recall the definitions of the four theta functions that can be found in Ref [1], chapter 21, namely

$$\theta_1(z;q) = \sum_{n=-\infty}^{+\infty} (-1)^{n-\frac{1}{2}} q^{\left(n+\frac{1}{2}\right)^2} \exp\left[(2\,n+1)\,iz\right], \qquad (1)$$

$$\theta_2(z;q) = \sum_{n=-\infty}^{+\infty} q^{\left(n+\frac{1}{2}\right)^2} \exp\left[(2\,n+1)\,iz\right], \qquad (2)$$

$$\theta_3(z;q) = \sum_{n=-\infty}^{+\infty} q^{n^2} \exp\left[(2\,n)\,iz\right], \qquad (3)$$

$$\theta_4(z;q) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} \exp\left[(2\,n)\,iz\right], \qquad (4)$$

After the above definitions, we are interested in finding explicit expressions for integrals of the kind

$$I_i \equiv \int_0^{+\infty} ds\, s^{-d} \exp\left(-c^2 s\right) \theta_i\left[abs; \exp\left(-a^2 s\right)\right], \ (a,b,c,d) \in \mathbb{C}, \qquad (5)$$

with $i = 1,2,3,4$.

For convenience, we start by $i = 2$ in Eq. (5). Using the definition as in Eq. (2), we have

$$I_2 = \int_0^{+\infty} ds\, s^{-d} \exp\left(-c^2 s\right) \sum_{n=-\infty}^{+\infty} \exp\left\{-a^2\, s\left[n^2 + 2n\left(\frac{1}{2} - i\frac{b}{a}\right)\right]\right\} \times \exp\left(iabs\right) \exp\left(-a^2 s/4\right). \qquad (6)$$

After completing the square in the last expression, we get,

$$I_2 = \int_0^{+\infty} ds\, s^{-d} \exp\left[-\left(b^2+c^2\right) s\right] \sum_{n=-\infty}^{+\infty} \exp\left[-a^2\, s\left(n-\tilde{b}\right)^2\right], \qquad (7)$$

where

$$\tilde{b} \equiv i\,\frac{b}{a} - \frac{1}{2}. \qquad (8)$$

Using the Poisson summation formula [19]

$$\sum_{n=-\infty}^{+\infty} \exp\left[-\sigma(n-\tilde{b})^2\right] = \sum_{n=-\infty}^{+\infty} \sqrt{\frac{\pi}{\sigma}} \exp\left[-\left(\frac{\pi^2 n^2}{\sigma}\right)\right] \exp\left(-2\pi i\tilde{b}n\right),$$

we have, after the change of variable $\sigma = a^2 s$ and taking into account Eq. (8),

$$\sum_{n=-\infty}^{+\infty} \exp\left[-a^2 s\left(n-\tilde{b}\right)^2\right] = s^{-1/2}\frac{\sqrt{\pi}}{|a|}\left\{1 + 2\sum_{n=1}^{+\infty} \exp\left[-\left(\frac{\pi^2 n^2}{a^2\, s}\right)\right] (-1)^n \cosh\left(\frac{2\pi b\, n}{a}\right)\right\} \qquad (9)$$

Now we can replace the right-hand side of Eq. (9) in Eq. (7):

$$I_2 = \frac{\sqrt{\pi}}{|a|}\int_0^{+\infty} ds\, s^{\left(\frac{1}{2}-d\right)-1} \exp\left[-\left(b^2+c^2\right)\, s\right] \times \left\{1 + 2\sum_{n=1}^{+\infty} \exp\left[-\left(\frac{\pi^2 n^2}{a^2\, s}\right)\right] (-1)^n \cosh\left(\frac{2\pi b\, n}{a}\right)\right\}. \qquad (10)$$

Remembering that the gamma function of argument ν and the modified Bessel function of the second kind of order ν, have the following representations [20]

$$\int_0^{+\infty} ds\, s^{\nu-1} \exp(-Bs) = B^{-\nu}\,\Gamma(\nu), \qquad (11)$$

$$\int_0^{+\infty} ds\, s^{\nu-1} \exp(-Bs) \exp(-A/s) = 2\left(\sqrt{A/B}\right)^{\nu} K_\nu\left(2\sqrt{AB}\right), \qquad (12)$$

the Eq. (10) reads

$$I_2 = \frac{\sqrt{\pi}}{|a|}\left\{\frac{\Gamma(1/2-d)}{\left(b^2+c^2\right)^{(1/2-d)}} + 4\sum_{n=1}^{+\infty} (-1)^n \cosh\left(\frac{2\pi b\, n}{a}\right) \left(\frac{\pi\, n}{|a|\sqrt{b^2+c^2}}\right)^{(1/2-d)} \times K_{1/2-d}\left(\frac{2\pi n\sqrt{b^2+c^2}}{|a|}\right)\right\} \qquad (13)$$

Carrying out steps completely similar to what we did for I_2, we find the expressions

$$I_1 = \frac{\sqrt{\pi}}{|a|}\left\{2\sum_{n=1}^{+\infty} (-1)^n \exp\left[-\frac{2\pi b}{a}\left(n-\frac{1}{2}\right)\right] \times \left(\frac{\pi}{|a|\sqrt{b^2+c^2}}\left|n-\frac{1}{2}\right|\right)^{(1/2-d)} K_{1/2-d}\left(\frac{2\pi\sqrt{b^2+c^2}}{|a|}\left|n-\frac{1}{2}\right|\right) + \right.$$
$$\left. + \ 2\sum_{n=0}^{+\infty} (-1)^n \exp\left[\frac{2\pi b}{a}\left(n+\frac{1}{2}\right)\right] \times \left[\frac{\pi}{|a|\sqrt{b^2+c^2}}\left(n+\frac{1}{2}\right)\right]^{(1/2-d)} K_{1/2-d}\left[\frac{2\pi\sqrt{b^2+c^2}}{|a|}\left(n+\frac{1}{2}\right)\right]\right\}. \qquad (14)$$

For $i = 3$, we obtain

$$I_3 = \frac{\sqrt{\pi}}{|a|}\left\{\frac{\Gamma(1/2-d)}{\left(b^2+c^2\right)^{(1/2-d)}} + 4\sum_{n=1}^{+\infty} \cosh\left(\frac{2\pi b\, n}{a}\right) \left(\frac{\pi\, n}{|a|\sqrt{b^2+c^2}}\right)^{(1/2-d)} \times K_{1/2-d}\left(\frac{2\pi n\sqrt{b^2+c^2}}{|a|}\right)\right\}, \qquad (15)$$

and finally

$$I_4 = \frac{\sqrt{\pi}}{|a|}\left\{2\sum_{n=1}^{+\infty}\exp\left[\frac{2\pi b}{a}\left(n-\frac{i}{2}\right)\right]\left(\frac{\pi}{|a|\sqrt{b^2+c^2}}\left|n-\frac{i}{2}\right|\right)^{(1/2-d)} \times K_{1/2-d}\left(\frac{2\pi\sqrt{b^2+c^2}}{|a|}\left|n-\frac{i}{2}\right|\right)+ + 2\sum_{n=0}^{+\infty}\exp\left[-\frac{2\pi b}{a}\left(n+\frac{i}{2}\right)\right]\left(\frac{\pi}{|a|\sqrt{b^2+c^2}}\left|n+\frac{i}{2}\right|\right)^{(1/2-d)} \times K_{1/2-d}\left(\frac{2\pi\sqrt{b^2+c^2}}{|a|}\left|n+\frac{i}{2}\right|\right)\right\}. \tag{16}$$

In the next section, we apply I_2 and I_3 to an interacting quantum gas of fermions and bosons, respectively.

Applications

In this Section, we will apply the results found previously to two different systems of interacting quantum particles: boson gas and fermion gas both defined on four-dimensional Euclidean space and under a constant magnetic field.

Quantum Boson Gas

Let us apply our findings to a boson system described by the scalar field ϕ under the interaction $(\lambda_0/4!)\phi^4$ in a four-dimensional Euclidean space. The Hamiltonian density is given by

$$\mathcal{H} = (\partial_\mu\phi)^2 + m_0^2\,\phi^2 + \frac{\lambda_0}{4!}\,\phi^4,$$

where m_0 is the bare-mass parameter and λ_0 is the self-coupling constant of the model both at zero temperature.

The interaction term $m_0^2\,\phi^2 + (\lambda_0/4!)\,\phi^4$ at zero temperature, has the symmetry $\phi \to -\phi$. However, as we shall see, at finite temperature T (such that $T < T$c, where Tc is the critical temperature of the boson gas), the system exhibits spontaneous symmetry breaking. Initially, we considered the interacting system in contact with the thermal reservoir, such that the corrections in one loop to the mass parameter are the type

$$m^2 = m_0^2 + \Sigma. \tag{17}$$

The correction Σ being given by (at zero temperature)

$$\Sigma = -\frac{\lambda_0}{2}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m_0^2)}, \tag{18}$$

with $p^2 = p_\tau^2 + p_x^2 + p_y^2 + p_z^2$. Through imaginary-time formalism, we introduce the thermal effects over the quantum system by replaces

$$\int\frac{dp_\tau}{2\pi} \to \frac{1}{\beta}\sum_{n_\tau} \quad ; \quad p_\tau \to \omega_{n_\tau} - i\mu, \quad \forall \quad n_\tau \in \mathbb{Z}, \tag{19}$$

where $\beta^{-1} = T$ is the temperature of the system, μ its chemical potential and $\omega_{n_\tau} = (2n_\tau)\pi/\beta$ are the Matsubara frequencies of the field ϕ. The even numbers in ω_{n_τ} come from the Kubo-Martin-Schwinger (KMS) conditions for bosons [21].

After using the identity

$$\mathcal{O}^{-1} = \int_0^{+\infty} ds\,\exp\left(-s\,\mathcal{O}\right)$$

and the substitutions stated in Eq. (19) in Eq. (18) for $\mathcal{O} \equiv p^2 + m_0^2$, we obtain

$$\Sigma(T) = -\frac{\lambda_0}{2\beta}\int_0^{+\infty} ds \int\frac{d^3p}{(2\pi)^3}\sum_{n_\tau=-\infty}^{+\infty}\exp\left\{-s\left[\left(\frac{\pi}{\beta}(2n_\tau) - i\mu\right)^2 + \vec{p}^{\,2} + m_0^2\right]\right\}. \tag{20}$$

After some manipulations and calculating three Gaussian integrals, we get

$$\Sigma(T) = -\frac{\lambda_0}{2\cdot 8\,\pi^{3/2}\,\beta}I_3, \tag{21}$$

where we used the Eq. (15) with the identification $a = 2\pi/\beta,\ b = \mu,\ c = (m_0^2 - \mu^2)^{1/2}$ and $d = 3/2$.

The free energy of the system is given by

$\mathcal{F} \equiv -m^2(\mathrm{T})\phi^2 + \lambda\phi^4$,
the minus sign is due to the system being in an ordered phase (broken symmetry) at zero temperature. We also define $\lambda = \lambda_0/(8\pi^{3/2})$.

Quantum Boson Gas under Magnetics Effects

Again, let us use Eq. (17) to perform the magnetic and thermal corrections on the mass parameter m_0^2. In this case, the Hamiltonian density at zero temperature is given by

$$\mathcal{H} = (D_\mu\phi)^2 + m_0^2\,\phi^2 + \frac{\lambda_0}{4!}\,\phi^4,$$

where $D_\mu = \partial_\mu - i\,e\,A_\mu^{ext}$ is the covariant derivative. We choose the Landau gauge: $A_\mu^{ext} : (0, 0, xB, 0)$. This gauge represents a constant magnetic field along the z direction.

In virtue of dimensional reduction $D \to (D - 2)$ consequence of magnetic field applied, we have [22]

$$\Sigma(\omega) = -\frac{\lambda_0}{2}\frac{\omega}{2\pi}\sum_{\ell=0}^{+\infty}\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_0^2)}, \tag{22}$$

being $p^2 = p_\tau^2 + p_z^2 + \omega(2\ell+1)$. Here, $\omega \equiv eB$ is the cyclotron frequency, and $\ell = 0,1,2, \dots$ labels the Landau levels.

Following analogous steps as done to obtain Eq. (21), the self-energy including both temperature and magnetic effects read

$$\Sigma(T,B) = -\frac{\lambda_0\,\omega}{8\,\pi^{3/2}\,\beta}\sum_{\ell=0}^{+\infty} I_3, \tag{23}$$

where we used the Eq. (15) with the identification

$a = 2\pi/\beta$, $b = \mu$, $c = \left[m_0^2 - \mu^2 + \omega(2\ell+1)\right]^{1/2}$ and $d = 1/2$.

The free energy of the system under magnetic effects is given by

$\mathcal{F} = -m^2(T,\omega)\phi^2 + \lambda\phi^4$.

We observe that the critical temperature is obtained when the boson system has zero mass parameter, i.e., $m(T_c,\omega) = 0$ (see Ref [23]).

For performing the phase structure analysis of the system we fix $m_\sigma = 2m_0$, being m_σ the scalar particle whose value used in this paper is $m_\sigma = 0.500$ GeV [24]. For the graphics, we used $\lambda = 0.17$.
In Figure 1, we show the behavior of the mass parameter m as a function of temperature for the boson gas in interaction under several values of an external magnetic field.

Fixing the value of the magnetic field, we see that the finite chemical potential tends to decrease the critical temperature of the system. That is, higher chemical potential values contribute to the system undergoing a phase transition at lower temperatures than in the case of zero chemical potential.

On the other hand, by fixing μ and taking different values of ω, the critical temperature of the system increases, for higher external magnetic field strengths. The effect of the external magnetic field on the system is to facilitate the phase transition. This is the phenomenon of magnetic catalysis for the charged scalar field.

To complete the analysis of the phase structure of the model, we investigate the free energy of the system in Figures 2 and 3. In Figure 2, we note that $T_c\ (\omega = 0) < T_c\ (\omega = 0.5)$. In particular, the temperature $T = 0.400$ GeV at which the system is in the symmetric phase for $\omega = 0$ GeV^2 becomes a temperature that guarantees symmetry breaking for $\omega = 0.5$ GeV^2. The same analysis is valid for Figure 3, however, the effect of $\mu = 0.200$ GeV is to attenuate the T_c value shown in Figure 2.

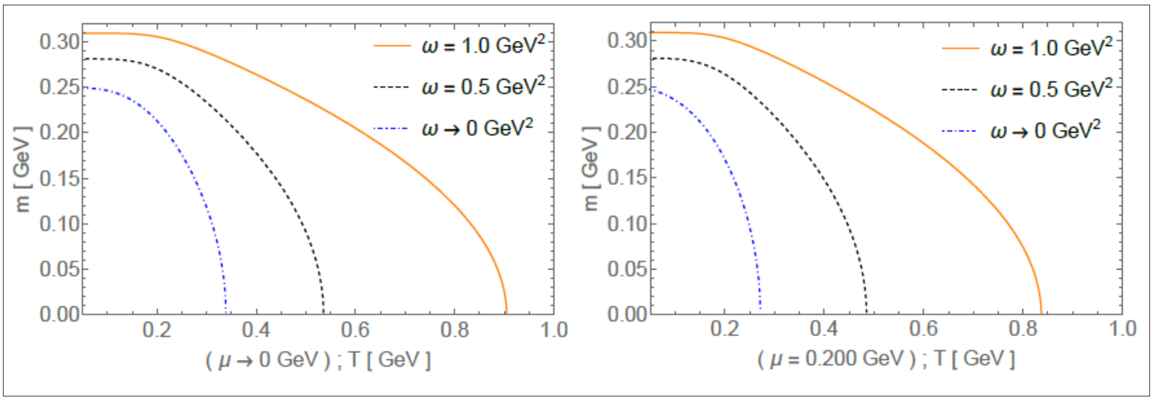

Figure 1: Mass parameter m as a function of temperature for zero (left plot) and non-zero (right plot) chemical potential and several values of the external magnetic field. We used, in [GeV]^2, $\omega \equiv eB = 1.0; 0.5; 0$ for full, dashed and dot-dashed curves, respectively. Colors at the online version.

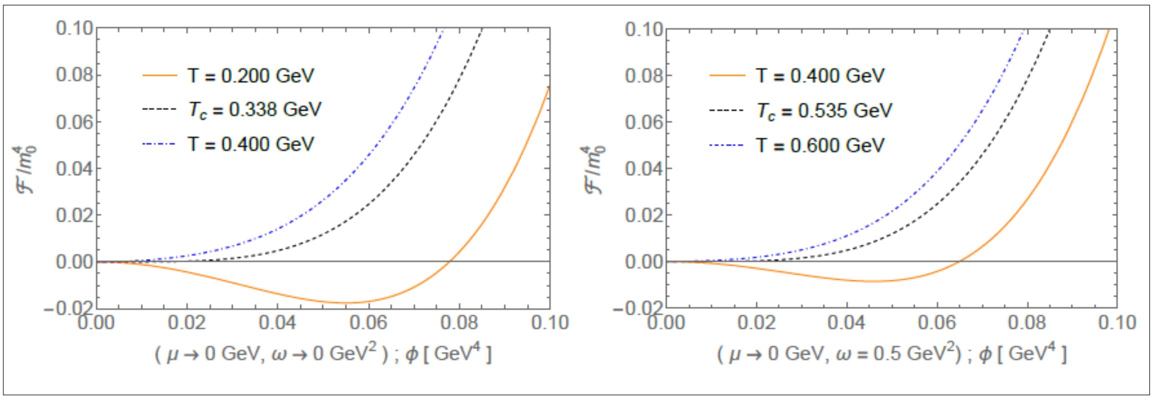

Figure 2: Free energy at zero chemical potential. On the left, we have zero magnetic field and on the right, we have a magnetic background. Note the increase of critical temperature, in [GeV], for augmentation magnetic field in [GeV]^2: $T_c\ (\omega = 0) < T_c\ (\omega = 0.5)$, i.e., the magnetic catalysis phenomenon. Colors at the online version.

Quantum Fermion Gas
Another application of our results arises when we consider a four-fermion interacting model type Nambu-Jona-Lasinio describing quarks with punctual interaction, whose density Hamiltonian is given by [14-16]

$$\mathcal{H}_{\mathrm{NJL}} = \bar{\psi}(\not{\partial} + m_0)\psi + G_s\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right], \quad (24)$$

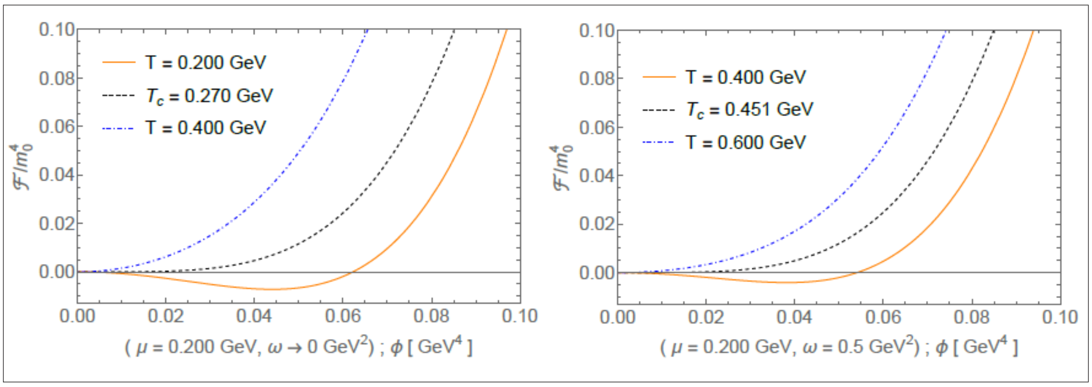

Figure 3: Free energy at a finite chemical potential. On the left, we have zero magnetic field and, on the right, we have a magnetic background. The present panels show attenuation of critical temperature values concerning Figure 2, due to $\mu = 0.200$ [GeV]. Again, we have the magnetic catalysis phenomenon. Colours at the online version.

where m_0 is the current mass quark and G_s correspond to self-coupling in the scalar channel. Using the mean field approximation, the gap equation reads

$$M = m_0 - 2\,G_s\,\Phi, \qquad (25)$$

being M the constituent quark mass (effective quark mass). The fermion condensate in the momenta space is

$$\Phi \equiv \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\,(S_F(p)), \qquad (26)$$

with $S_F(p) = (\not{p} - M)/(p^2 + M^2)$ representing the fermion kernel propagator and $p^2 = p_\tau^2 + \vec{p}^{\,2}$. Note that $\not{p} = \gamma_\alpha p_\alpha$, γ_α being the Dirac matrices in the chiral representation. The symbol Tr means the trace over Dirac matrices, flavor, and color spaces. Since the trace of an odd number of γ- matrices vanishes, we obtain

$$\Phi = -4MN_c\,N_f \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+M^2}. \qquad (27)$$

Let us fix the fermion field carrying $N_f = 2$ flavors with $N_c = 3$ colors.

To include thermal effects on the model we use the Matsubara prescription for fermions

$$\int \frac{dp_\tau}{2\pi} \to \frac{1}{\beta}\sum_{n_\tau} \;\; ; \;\; p_\tau \to \omega_{n_\tau} - i\mu, \quad \forall \quad n_\tau \in \mathbb{Z}. \qquad (28)$$

Again, $\beta^{-1} = T$ is the temperature of the system, μ its chemical potential and $\omega_{n_\tau} = (2n_\tau + 1)\pi/\beta$ are the Matsubara frequencies of the fermionic field. The odd numbers in ω_{n_τ} are due to the KMS conditions for fermions.

Then, after analogous steps to those that led to Eq. (21), the chiral condensate under chemical potential and temperature effects are written as

$$\Phi(T) \;=\; -4MN_c\,N_f\frac{1}{8\,\pi^{3/2}\,\beta} \times \int_0^{+\infty} ds\, s^{-3/2}\,\exp\left[-\left(M^2-\mu^2\right)s\right]\,\theta_2\left[\frac{2\pi\mu s}{\beta};\exp\left(-4\pi^2 s/\beta^2\right)\right]. \qquad (29)$$

Through identification $a = 2\pi/\beta$, $b = \mu$, $c = (M^2 - \mu^2)^{1/2}$ and $d = 3/2$, the quark condensate is expressed in terms of I_2 given by Eq. (13), namely

$$\Phi(T) \;=\; -4MN_c\,N_f\frac{1}{8\,\pi^{3/2}\,\beta}I_2. \qquad (30)$$

Quantum Fermion Gas under Magnetic effects

Under a magnetic field in the z-direction, the Hamiltonian density becomes

$$\mathcal{H}_{\rm NJL} = \bar{\psi}(\not{D} + m_0)\psi + G_s\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right], \qquad (31)$$

again $D_\mu = \partial_\mu - i\,Q_f\,A_\mu^{ext}$, for $Q_u = (2/3)e$ e $|Q_d| = (1/3)e$, moreover, the gauge choice is the same as for the scalar case.

The chiral condensate including magnetic effects can be written as

$$\Phi(B) = -4MN_c\sum_{f=u}^{d}\left(\frac{\omega_f}{2\pi}\right)\sum_{\ell=0}^{+\infty}\sum_{\sigma=\pm1}\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+M^2}, \qquad (32)$$

being $p^2 = p_\tau^2 + p_z^2 + \omega_f(2\ell + 1 + \sigma)$, such that $\omega_f \equiv |Q_f|B$ is the cyclotron frequency, $\sigma = \pm 1$ represents the spin of the fermionic field and $\ell = 0,1,2,\ldots$, denotes the Landau levels.

After including thermal effects and following steps similar to demonstration Eq. (21), the quark condensate reads

$$\Phi(T,B) = -\frac{4MN_c}{2\sqrt{\pi}\,\beta}\sum_{f=u}^{d}\left(\frac{\omega_f}{2\pi}\right)\sum_{\ell=0}^{+\infty}\sum_{\sigma=\pm1} I_2, \qquad (33)$$

where I_2 is written in the Eq. (13), for $a = 2\pi/\beta$, $b = \mu$, $c = [M^2 - \mu^2 + \omega_f(2\ell + 1 + \sigma)]^{1/2}$ and $d = 1/2$. Now, we can use Eq. (25) for describing the constituent quark mass taking into account thermal and magnetic effects.

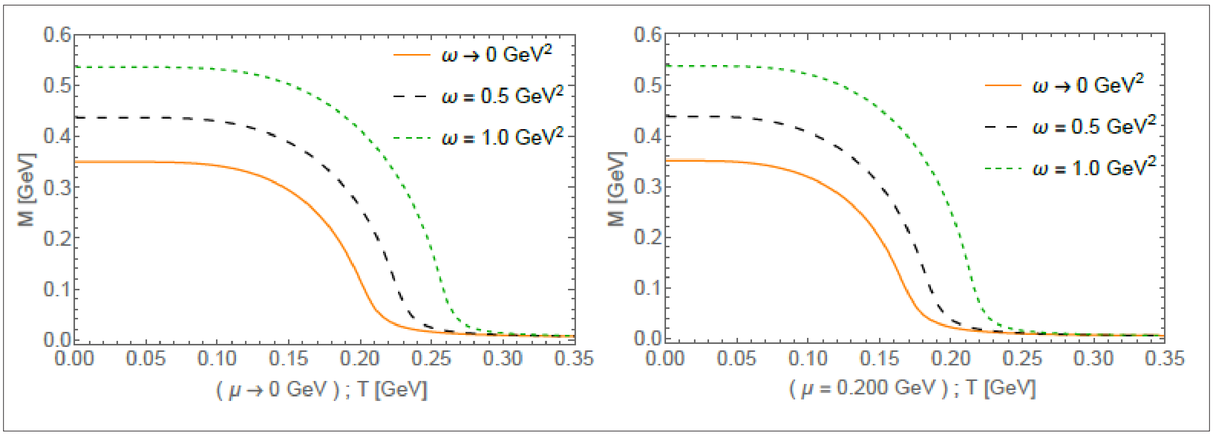

Figure 4: Constituent quark mass as a function of temperature for zero (left plot) and finite (right plot) chemical potential and several values of an external magnetic field. We have used, in $[\text{GeV}]^2$, $\omega \equiv eB = 0; 0.5; 1.0$ for full, dashed (large) and dashed (small) curves, respectively. Colors at the online version.

In the Figures. 4 and 5 are analyzed the gap equation and thermal mass gradient of the fermion model. We use the parameters that fit the mass and pion decay constant in vacuum, namely $m_\pi = 0.138$ GeV and $f_\pi = 0.092$ GeV:

$$m_0 \;=\; 0.005\,\text{GeV}\;;\; G_s \;=\; 4.730\,\text{GeV}^{-2}\;;\; M_0 = 0.350\,\text{GeV}.$$

In Figure 4 we note that the chemical potential has little influence on the fermionic system. Furthermore, for temperatures close to zero, the system does not depend on the chemical potential, since the constituent quark mass presents practically the same values for the fixed magnetic field.

In Figure 5 we plot the thermal mass gradient. The peak of the curve indicates the temperature at which the chiral phase transition occurs [25]. As in the bosonic case, the finite chemical potential tends to lower the transition temperature of the system. Still in Figure 5, but considering the dependence on the external field, we can observe that the chiral transition is stimulated by increasing magnetic fields. Thus, we have the magnetic catalysis phenomenon again, now for fermions.

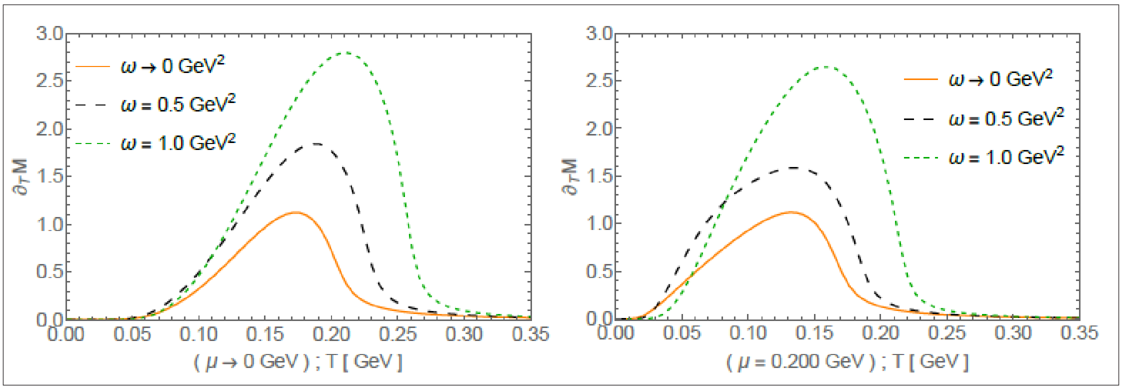

Figure 5: The thermal mass gradient versus temperature at zero (left plot) and non-zero (right plot) chemical potential and the same values of the external magnetic field showed in Figure 4. We have $T_c\ (\omega = 0) < T_c\ (\omega \neq 0)$, according with the magnetic catalysis. Colors at the online version.

Conclusion

In this paper, we study four integrals of Jacobi's theta functions. By the Poisson sum formula and the integral representations of the modified Bessel function of the second kind and gamma function, we were able to express the solutions of I_i, with $i = 1,2,3,4$ in terms of K_ν. We also apply the expressions found in a quantum gas of bosons and in a quantum gas of fermions, both self-interacting and under an external magnetic field. Due to KMS conditions, we used periodic boundary conditions in imaginary-time coordinates for bosons, which meant θ_3 in the heated gas. On the other hand, the KMS conditions applied to the fermions meant antiperiodic boundary conditions in the imaginary-time coordinates of this system. In this last case, we express the thermal effects by functions θ_2.

From the phase structure of the systems, we observe the same behaviour of the parameters of mass (bosons) and constituent mass of quarks (fermions) for chemical potential zero and different from zero, namely: finite chemical potential makes the transition temperatures lower. In contrast, the effect due to the magnetic background is to increase the transition temperatures as the external field increases. This is the magnetic catalysis phenomenon.

Another point to highlight is in relation to the first term of Eqs. (13) and (15). These terms have a singularity of the type: $\Gamma(\frac{1}{2}-d)/(\mathcal{M}^2)^{\frac{1}{2}-d}$, which is divergent for $d = 3/2$ or $d = 1/2$ as used in this note. For this reason, we evaluated it numerically.

Jacobi theta functions are an interesting research topic with great potential for application in QM and QFT. We look forward to continuing to work on these issues in other models and backgrounds.

Use of AI tools declaration
The authors declare they have not used Artificial Intelligence (AI) tools in the creation
of this article.

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Conflict of interest
The authors declare there is no conflict of interest.

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