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Excel Files for Teaching Caustics of Rainbow and Lenses

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ABSTRACT

In this paper, Software is presented for teaching through interactive demonstrations about caustics on rainbows and lenses. At first, we explore the caustics on rainbows since they are amenable to analytic calculations. We find 4 kinds of caustics: 2 internal (of first refraction and reflection) and 2 externals (second refraction and exciting from the back of the drop, and 2nd refraction emerging on the front). We examine caustics of lenses constructed by two spherical surfaces. We explore the ray diagrams and wave fronts and give methods for finding the caustics. We also examine the case of thick lens and the caustics due to reflection.

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Caustics for Rainbow and Lenses

Caustics is an interesting phenomenon for the teaching of optics. The use of caustics due to reflection were are given ways for explanation, experimentation and prediction [1].

Historical Facts about the Rainbow

Explanations of the rainbow as due to reflection can be found in Aristotle, Al Haytham [2, 3, 4]. Avicenna and Al Farisi and Theodoric used refraction for explaining the rainbow [5]. Al Haytham's explanation was based on his work on convex mirrors. He considered the cloud as having a convex spherical form [4].

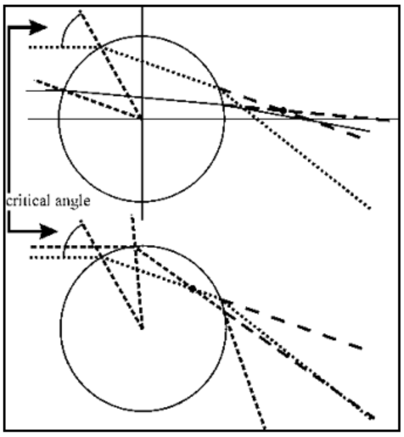
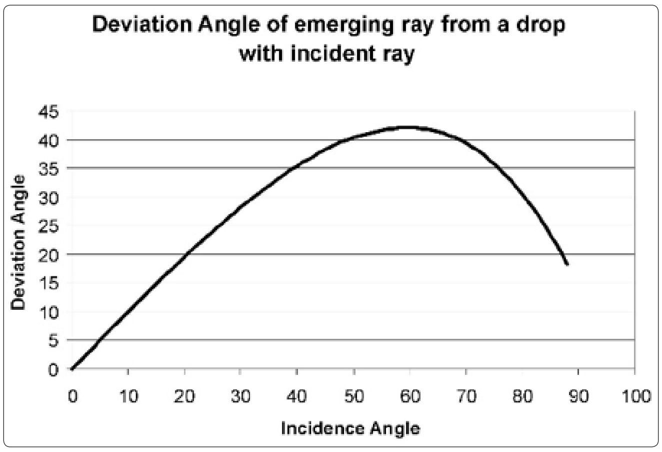


Figure 1: Relation of rays to the special ray of critical angle

Usually the explanation of the rainbow is attributed to Descartes [6]. But actually Kamal al Din Al Farisi and Theodoric [5, 7, 8] developed the ideas of the Rainbow, and did experiments with spherical bottles. Boyer considers Al Farisi as getting his ideas from his teacher Qutb al Din al Shirazi. According to Rashed al Shirazi gave to his student the task of studying the rainbow and gave him Al Haytham's manuscripts. Al Haytham had derived a rule about the two refractions of light as it passes through a sphere: He considered the light ray incident parallel to the axis OO'. The trajectory that the light follows between two refractions are such that as the angle of incidence increases, the distance of the point where the light meets the sphere after its first refraction from the axis increases until the angle of incidence reaches a "critical" value. After this critical value the distance decreases. Al Farisi described the rays as belonging to the "central cone" if their angle of incidence is less than the critical value, and as belonging to the "external cone" if their angle of incidence is bigger than the critical value. Another property of the ray following the critical angle is that the rays of the internal cone are cut by the critical ray outside the sphere, while the rays of the external cone are cut inside the sphere. In the first case the angle of the rays, which is equal to the difference of the deviation angles, is less than half the difference of angles of incidence, while on the second case the angle of rays are more than half the angles of incidence. By taking the limiting value of a ray, for example of the external cone approaching the critical ray, the angle of difference becomes ½ the difference of angles of incidence. This way we have:



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$$\lim_{\theta_i \to \theta_c}\left(\arcsin\left(\frac{\sin(\theta_i)}{n}\right) - \arcsin\left(\frac{\sin(\theta_c)}{n}\right)\right) = \frac{1}{2}\lim_{\theta_i \to \theta_c}(\theta_i - \theta_c)$$

After some manipulations we get: $\sin(\theta_c) = \sqrt{\frac{4-n^2}{3}}$, from which the critical angle for water is θ_c=59,58°. This special angle of incidence was later discovered again by Descartes, who according to Boyer must had done laborious calculations and patient observations, applied to large numbers of individual rays [3].

The light after reaching the surface of the sphere either is reflected or refracted. In the first case we have a "zero order" rainbow, which is unobservable for parallel light rays (Look &McCollum 1994). If the light is reflected once and then refracted to the air, we have a "first order" rainbow. This ray corresponds to the "Cartesian" ray (Walker 1976). For more reflections we have higher order rainbows. . It can be seen that on the formation of the rainbow, the most important rays are that of the external cone. If as deviation angle for the rainbow is defined the angle between the directions of the incident ray and the emergent ray, then it can be seen that only for the first order rainbow, the deviation is minimum for the special ray of Al Farisi. For the zero order it can be seen that the deviation angle is a monotonic function. A very simple simulation can be constructed using Excel for the first order rainbow. In this case the deviation angle presents a maximum. The students feel a satisfaction when they see by themselves that they produce a kind of representation of the rainbow. Students also can make another representation which is seen in figure 15. Descartes and Al Farisi's theory was good for explaining the shape of the rainbow. Al Farisi's theory of colors was very similar to Aristotle's theory as it is expounded in his Meteorologica [2].

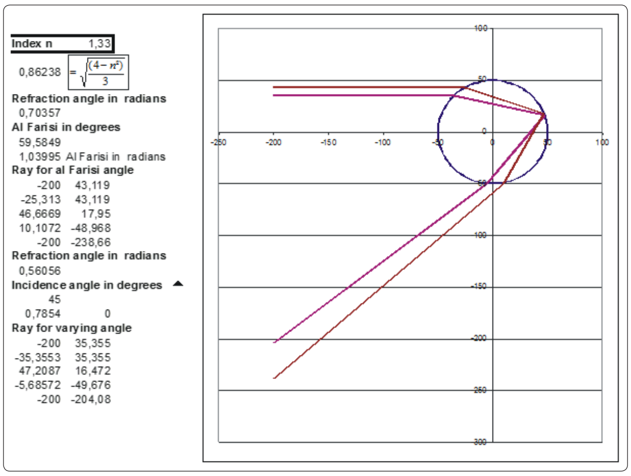

Figure 3: An Excel simulation for students

Caustics

After the consideration of rainbow we turn to caustics. The rainbow caustics can be used as an introduction of caustics, Rainbow caustics were studied mathematically by R.Potter [9, 10]. On the same volume is contained also Airy's theory of the rainbow. Some corrections on Potter's considerations were reported by [11].

Rainbow Caustics

The Descartes ray is a first step on rainbow caustics. We can study here 4 caustics for rainbow:

a) A caustic due to the emerging rays on the back of the drop of water (DIACAUSTIC).
b) A caustic from the points where the rays inside the drop meet
c) A caustic for the reflected rays (internal caustic)
d) A caustic for the rays that emerge after one refraction.

In the case of a parallel beam the calculations for the rays (refracted, reflected etc.) is elementary.

Diacaustic and refraction on a circle If a ray is parallel to the x-axis then the angle of incidence is φ and the angle of refraction θ. $\varphi = asin(y/R)$

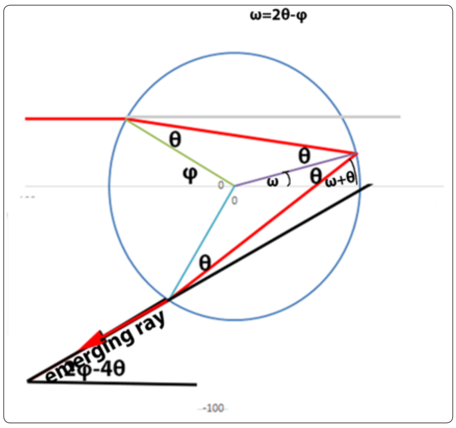

Figure 4: Angle of incidence φ angle of refraction θ, inclination of emerging reflected ray 2φ-4θ

Internal Caustic from Refraction

If the angle of incidence is φ then the ray from the point of incidence has an inclination *klisi*= φ-ϑ where ϑ is the angle of refraction (figure 3). The mathematics of the simple model are elementary and can be followed by the students.

$$\frac{d\theta}{d\varphi} = 1/\left(1 - \frac{\sin^2\varphi}{n^2}\right)^{\frac{1}{2}} * \frac{cos\varphi}{n}$$

$$x_{caustic} = -\frac{R\left(cos\theta \cdot \cos(\theta-\varphi) + \cos\varphi\left(\frac{d\theta}{d\varphi} - 1\right)\right)}{\frac{d\theta}{d\varphi} - 1}$$ is the coordinate of the caustic

Internal and External Caustic from Reflection

In this case the inclination we put klisi = $tan(\omega+\vartheta)$ and

$$\frac{dklisi}{d\varphi} = \frac{1}{\cos^2(\omega+\theta)}\left(\frac{d\omega}{d\varphi} + \frac{d\theta}{d\varphi}\right) = \frac{1}{\cos^2(\omega+\theta)}\left(\frac{3n \cdot cos\varphi}{\sqrt{n^2 - sin^2\varphi}} - 1\right)$$

For dklisi/dφ = 0 $\frac{3n \cdot co\varphi}{\sqrt{n^2 - sin^2\varphi}} - 1 = 0 \ or$

$$9n^2 \cdot \cos^2\varphi = n^2 - 1 + \cos^2\varphi \ or \ \cos^2\varphi = \frac{n^2-1}{9n^2-1}$$

$cos\varphi = \frac{\sqrt{n^2-1}}{\sqrt{9n^2-1}}$ so for n=1,33 φ=1,27413 (73°)

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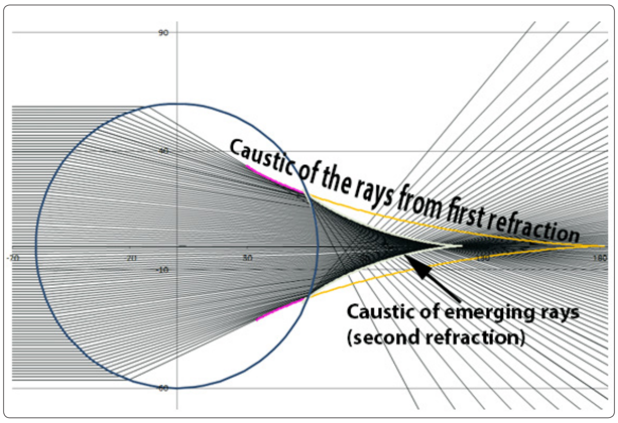

Figure 5: Caustics of refracted rays (zero order rainbow). The caustic of the refracted rays is given by the first refraction and by the second refraction

In case of one reflection we have two internal rays. The first has as inclination *φ-ϑ* και the 2nd inclination *ω+ϑ* while the emerging ray has inclination *D =4ϑ-2φ*
For the ray with inclination *ω+ϑ* $y=y_2+tan(\omega+\vartheta)*(x-x_2)$ so

$$\frac{dy2}{d\varphi}+\frac{1}{cos^2(\omega+\theta)}*\left(\frac{d\omega}{d\varphi}+\frac{d\theta}{d\varphi}\right)*(x-x_2)-tan(\omega+\theta)\cdot\frac{dx2}{d\varphi}=0$$

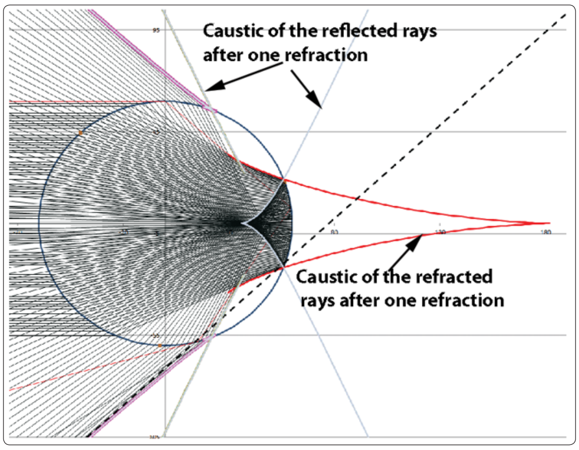

Figure 6: Caustics after one refraction

$$x=R\cdot\cos\omega-R\cdot\frac{cos\theta\cdot cos(\omega+\theta))\frac{d\omega}{d\varphi}}{\left(\frac{d\omega}{d\varphi}+\frac{d\theta}{d\varphi}\right)}$$

$y=y3+tan(2\varphi-4\theta)(xc-x3)$

$x_3=-R\cdot\cos(4\theta-\varphi)$

By substituting in the above relation for $x_{caustic}$ we get

$$x_{caustic}=-R\cdot(cos(\varphi-4\theta)-\frac{\cos(\varphi)\cos(4\theta-\varphi)}{2\left(2\cdot\frac{d\theta}{d\varphi}-1\right)}\left(4\cdot\frac{d\theta}{d\varphi}-1\right))$$

For x having an infinity value $2\cdot\frac{d\theta}{d\varphi}-1=0$ or

$$\frac{1}{\left(1-\frac{\sin^2\varphi}{n^2}\right)^{\frac{1}{2}}}*\frac{cos\varphi}{n}=\frac{1}{2}\ or\ 4\cos^2\varphi=(n^2-\sin^2\varphi)\ or\ 4-n^2=3\cdot\sin^2\varphi\ \varphi=\text{asin}\left(\left(\frac{4-n^2}{3}\right)^{\frac{1}{2}}\right)$$

which is the same value as the Cartesian Angle (or Al Farisi angle)

As we observe in the figure the reflected wave fronts have a characteristic point where there is an angle.

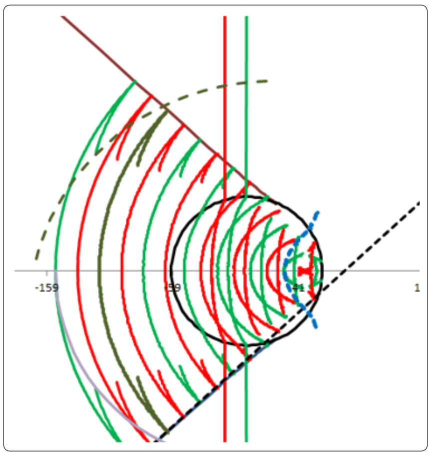

Figure 7: Caustic of the emerging rays and the wave fronts. The wave frons with vertical lines were not yet reflected. Also we see the "wave front" basic for Airy function

Teaching about Caustics and Lenses

The teaching of lenses can be facilitated when the student is having an image of what is being taught. The teaching of the theory of real lenses is not so easy to follow (Walther, 1996). Students can see the limitations of paraxial optics and very thin lenses. This spreadsheet permits to the student to use 5 kinds of simulation: a) Choose of radii and center of the circle on right. The program does ray tracing of the constructed lens. Except of the ray tracing, the student can observe the wave fronts b) Paraxial theory of lenses. c) Finding the image at the point of the maximum concentration of rays which is the cusp point of the caustic. d) Experiments for finding the focal length of a lens. e) Lenses that change shape when we change the position of the source or of the image.

As it is depicted in figure 8 there are two caustics. A) Caustic of the part of rays due to one refraction B) Caustic of rays after the second refraction (emerging rays). Other caustics can be found by calculating the caustics of reflected rays. The caustic of emerging rays can be used for the simulation of experiments of finding the focal length [12]. In this paper the simulation of the experiments is based on finding the location of the least deviation between the rays, while in this proposal instead of finding this position we determine the cusp of the caustic of the emerging rays.

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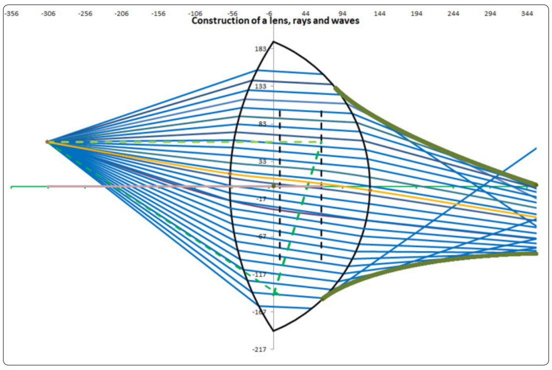

Figure 8: Caustics of the emerging rays and the refracted rays inside the lens

In the case of virtual image then the program gives the cut of the back continuation of the emerging rays.

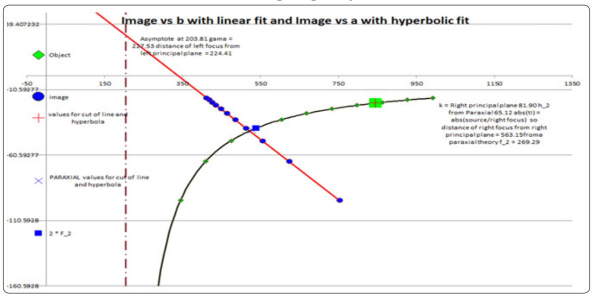

Figure 9: Finding the focal length

continuation of the emerging rays. While Newell and Baez give the caustic for the reflected rays we get the case of refraction as in the case of figure 10 we get the position of the virtual image, but also the wave fronts with the characteristic angles [13]. In figure 10 we can see the wave fronts having the characteristic breaking with the extreme points being on the caustic as in the case of the rainbow. This can be one method of finding the location of the caustic: locate the extreme points of the wave front.

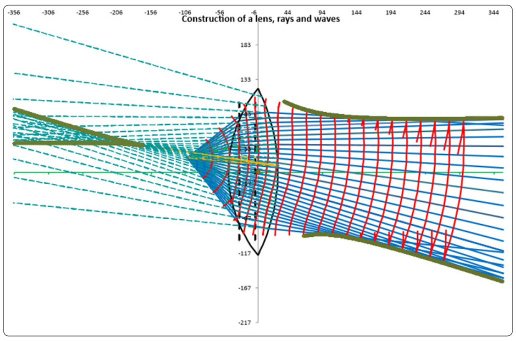

Figure 10: Case of virtual image. There are 3 branches of caustic

Caustics of Reflected Rays

There are interesting results in case of reflected rays. Reflected rays can have similar cases as in the case of Rainbows where the primary rainbow is due to one reflection. In the case of lenses we can have cases of one reflection, but also of multiple reflections on the back side of the lens.

The reflected rays show a case of critical behavior when the right lens is wide. In this case we may have multiple reflections. We have chosen the point where to the two circles meet is the point $(0, Y_{pano})$ where Y_{pano} is the highest point. The left side of the lens is a part of circle with radius R_B and center X_B and the right circle has radius R_A and center at X_A. If the point where the refracted ray from the first refraction will have coordinates (x_R, y_R) and inclination t. Then if the coordinates of the next point at the right side will be

$$x = x_R - \frac{2(x_R - x_A + y_R * t)}{1+t^2} \quad y = y_R + t\left(-\frac{2(x_R - x_A + y_R * t)}{1+t^2}\right)$$

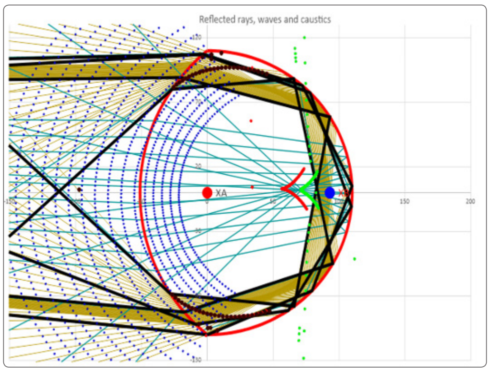

Figure 11: Caustics for reflection. We can see that the caustic of emerging rays has the same direction as the caustic of the reflected rays inside the lens

When x<0 then the reflected ray meets the left circle otherwise there will be a new reflection. We can see that the position of the different caustics is changing according to the radius of the circles.

In case the two circles are semicircles the caustics are the same as for the case of the rainbow.

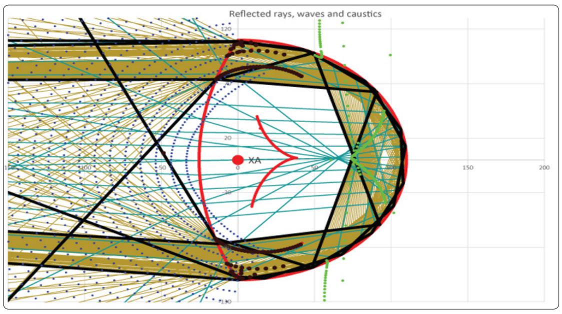

Figure 12: In this case the caustic of the emerging rays is opposite in direction opposite to the caustic of the rays reflected once

Huygens Lens

Huygen's presented in his work "TRAITE DU LUMIERE" a method for constructing a lens which will be capable of focusing perfectly the rays of light emanating from one point to another. In this paper we present

- The theory of finding the back surface of the lens.
- Closed Huygens Lens
- Moving the source from the initial point on the x-axis.
- Calculation of the caustic.

The Theory of Finding the Back Surface of the Lens

Let us assume that the source of light is located on the left of the lens and the focus on the right.

To construct the lens, we employ the Fermat's principle in the form of the time K needed for a ray

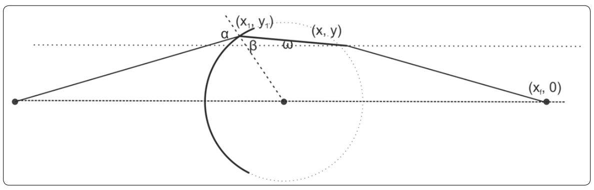

Figure 13: Geometrical considerations of a Huygens' Lens

move from the source through the lens and then to the focus is constant.

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If x_0 is the source point, x_1 the point on a spherical surface, x the point on the back of the lens and x_f is the focus point then the total time from x0 to xf is constant. So if $n_0=1$, n, n_2 are the indexes of refraction on the left (on the medium of the source), in the index of refraction inside the lens and n_2 the index of refraction on the right (where the focus f is located) and c is the

speed of the wave then $T = c\left(\frac{|\overrightarrow{x_0 - x_1}|}{1} + \frac{|\overrightarrow{x_1 - x}|}{n} + \frac{|\overrightarrow{x - x_f}|}{n_2}\right) = K$

or if c is the speed of light outside the lens and c/n is the speed of light inside the lens (where n is the index of refraction of the lens) then if $D_1 = |\overrightarrow{x_0 - x_1}|$ x is the distance of the source to the point on the circle, D_2 is the distance from the point on the circle to the back of the lens and D_3 is the distance from the point on the back of the lens to the focus then:

$$D_1 + n \bullet D_2 + n_2 D_3 = \text{constant} \qquad (1)$$

For a given incident ray we can calculate the x_1, y_1, D_1 and then to find the point of the pack of the lens (x,y) we use (1) as $n \bullet D_2 + n_2 \bullet D_3 = L$ where $L = K - \sqrt{(x_1 - x_0)^2 + y_1^2}$ is already known.

We express the above equation as

$$n \cdot \sqrt{(x_1 - x)^2 + (y_1 - y)^2} + n_2\sqrt{(x_f - x)^2 + y^2} = L$$

We put $A = (x_1 - x)^2 + (y_1 - y)^2$ and $B = (x_f - x)^2 + y^2$ Let us move the origin to the focus point then: xf = 0. (We have the transformation x' =x-xf so the following relations are done with this transformation so x_1' is x_1-x_f x'=x=x_f)

$$n \cdot \sqrt{A} + n_2\sqrt{B} = L(x, y) \qquad (2)$$

$$A = (x'_1 - x')^2 + (y_1 - y)^2 = (x'_1 - x')^2(1 + tan^2\omega) = \frac{(x'_1 - x')^2}{cos^2\omega}$$

$$B = (\xi + x'_1)^2 + (y - y_1 + y_1)^2 = {x'_1}^2 + ``{y_1}^2 + 2(x'_1 + y_1 tan\omega)\xi + \frac{\xi^2}{cos^2\omega}$$

Where x'-x'$_1$ = ξ . Equation 2 is written as aa quadratic equation. We put: $\varepsilon = x'_1 + y_1\ tan\omega$, $\delta = {x'_1}^2 + {y_1}^2$:

$$(n^2 - {n_2}^2) \cdot \frac{\xi^2}{cos^2\omega} - 2(L \cdot n + {n_2}^2 \cdot \varepsilon \cdot cos\omega)\frac{\xi}{cos\omega} + L^2 - {n_2}^2\delta = 0$$

So we find $\xi = x' - x'_1$ $x' = x'_1 + \xi$ and the position $x = x' + x_f$. Gonzalez_ Acuna and Chaparro_Romo have given a general formula for finding the shape and expanded this to different cases a rigorous analytical solution for the bi-aspheric singlet lens design problem and their formula can be analyzed in our Excel Sheet [14].

Wave fronts: We get the wave fronts starting at the source and ending on the focal point.

HU YGENS' LENS WITH INTERSECTION OF BACK AND FRONT SURFACES AT (0, y)

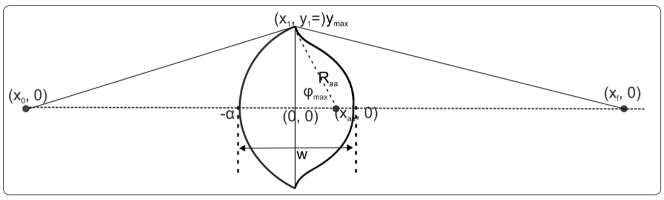

Figure 14: Geometry of a closed Huygens' Lens

Huygens lens we calculate is not a closed. The back surface can be any wavefront which will not close the lens at the highest and lowest points of the lens. When we impose the condition that the back and the front surfaces meet at x=0 then we have a modified behavior of the spherical surface on the front. Now we get a more classical "Huygens Lens" (figure 14)

To draw the rays in the two lenses we use the parametric equations of the circle or the hyperbola depending on the form of the front surface.

If the total time is L then for the highest point a is fixed and also the points x_0 and x_f then for the highest point:

$$L = \sqrt{(x_1 - x_0)^2 + y_1^2} + n_2\sqrt{\left(x_f - x_1\right)^2 + y_1^2}$$

this equation we can find y1 R_{aa}, x_{aa} From the solution we find the value of y which will be the highest point of the lens, which we call $y_{circmeg}$. Then the Radius of the spherical surface is found from $Raa = \frac{\left(y_{cicmeg}^2 + AKTINA^2\right)}{2 * AKTINA}$ where – AKTINA is the point on *x-axis* of the spherical surface. So the center of the circle is at $-AKTINA + R_{aa}$, and the maximum angle that the radius makes with *x-axis* is $f_{megy} = arcsin\left(\frac{y_{cicmeg}}{R_{aa}}\right)$

Hyperbolic Frontal Surface

For the case of hyperbola the finding of the maximum of y is the same as for the spherical case. The parametric equation of the hyperbola $x = -2 * a + \frac{ach}{cost}, y = b * tant$ gives for the

$atan\left(\frac{y_{max}}{b}\right)$ and to be on the x=0 we have ach = $2 * a * \cos(t_{tmax})$

Finding the Rays for the Case of the Source not lying on the X axis.

In this case the finding of the total ray path demands the finding of the refracted ray on the frontal surface.

To find the point (x_1, y_1) on the frontal surface for a ray starting from (x_0, y_0) (source) and passing through (x_2, y_2) on the back surface we can use Fermat's principle or solving the Snell's law for the (x1, y1).

We need to find the time for the straight line (x_0, y_0) -$(x_1$, y1) and we try all the possible times for the parameter φ taking values from φ_{max} to $-\varphi_{max}$.

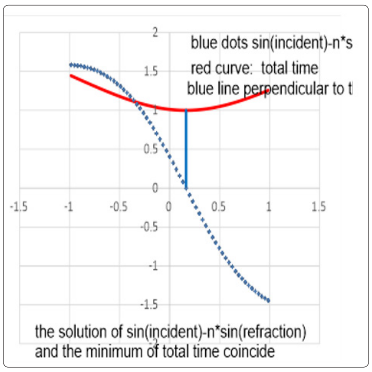

Figure 15: In the case of spherical surface we find the minimum of the total time and the law where we have the total time a minimum. We can see they coincide

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For the case of spherical frontal surface the finding of x_1, y_1. For $x_1 = x_{aa} - Raa\ \cos\varphi$ and $y_1 = Raa\ \sin\varphi$ and

$$\frac{dy}{dx} = \frac{1}{\tan\varphi}\ k = -\frac{1}{\frac{dy}{dx}} = -\tan\varphi\ k_x = \frac{1}{(1+\tan\varphi^2)^{\frac{1}{2}}} = \cos\varphi\ k_y = -\sin\varphi$$

The cut of the line from (x_0, y_0) to (x_2, y_2) cuts the circle at

(x_1, y_1) and has a $t = \frac{y_2 - y_0}{x_2 - x_0}$ so the line has equation

$y = y_0 + t * (x - x_0)$ circle: $(x - x_{aa})^2 + y^2 = R_{aa}^{\ 2}$ so the equation for the cut is: $x^2 - 2x\, x_{aa} + x^2_{aa} + R^2_{aa}\,(1+t^2) + 2x_{aa}\, x_0\, t^2 - 2x_{aa}\, y_0\, t - t^2 x^2_{aa} - (y_0 - x_0 t)^2$ Determinent $= R^2_{aa}(1+t^2) + 2x_{aa}\, x_0\, t^2 - 2x_{aa}\, y_0\, t - t^2 x^2_{aa} - (y_0 - x_0\, t)^2$ $x_1 = \frac{x_{aa} + t^2 x_0 - t * y\_0 \pm (Determinent)^{\frac{1}{2}}}{1+t^2}$

Hyperbolic Front Surface

For the hyperbolic frontal surface we use the parameter φ for expressing the hyperbola through $x_1 = 2a\text{-}ach/\cos\varphi$ and $y_1 =$ ba

$\tan\varphi$ Then $\frac{dy}{dx} = \frac{BA}{ach\ \sin\varphi}$ $\quad k = -\frac{1}{\frac{dy}{dx}} = -ach\frac{\sin\varphi}{ba}\ k_x =$

$$\frac{ba}{(ba^2 + ach^2\sin\varphi^2)^{\frac{1}{2}}} = k_y = \frac{-ach\ \sin\varphi}{(ba^2 + ach^2\sin\varphi^2)^{\frac{1}{2}}}$$

$$y = b\tan\varphi = y_0 + t\,x - t\,x_0 = y_0 - 2t\,a + \frac{a_h t}{\cos\varphi} - t x_0$$

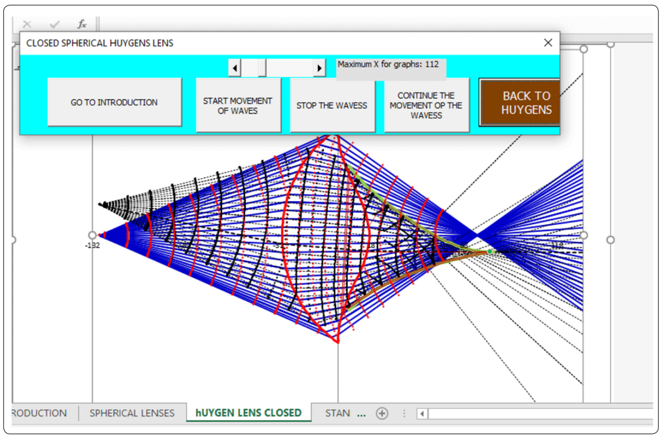
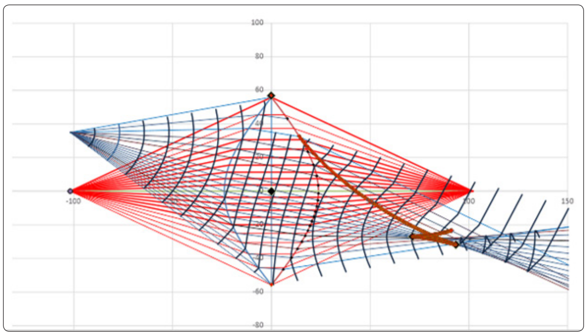


Figure 16: Huygens Lens with front surface spherical and back surface meeting in x=0. In this case the front circle has a movable center. The caustic for the lens can be found from the maxima or by using the formula given in the spherical lens

This permits to use a similar method to calculate the rays. The rays from an object that is located at the point (x0, y0) are calculated for both the lens with a spherical surface in front and a hyperbolic surface in front by the same method. We choose a point (x2, y2) on the back surface of the lens and we try to find a point (x1, y1) on the method. We choose a point (x_2, y_2) on the back surface of the lens and we try to find a point (x_1, y_1) on the front surface for which the time from the source (x_0, y_0) to (x_1, y_1) and then to (x_2, y_2) is the least. This is accomplished by a "user's" function. The perpendicular to this point (x_2, y_2) is determined by another "user's" function and then the continuation of the rays to a distant point (not the "focus"). On the basis of the rays we determine the wave fronts.

IBN SAHL'S LENS:

The case of a hyperbolic front surface has an important historical case when the back surface is a hyperbola of the same shape as the front case. In this case it is a Ibn Sahl lens if the source is on the left focus point of the left surface and the focus point is on the right focus point of the right hyperbola.



The lens has a maximum ymax and the source is at the focus of the left parabola. The maximum angle that a ray makes with the horizontal axis is $\theta_{max} = \text{atan}\left(\frac{y_{max}}{b}\right)$ since $y = b * \tan\theta$. The left side is the hyperbola x=a/cosθ moved by $\Delta x = -\frac{a}{\cos(\theta_{max})} - c = -d - c$ =SAHL which is also the point of the source on the left.

$\frac{dx}{d\theta} = \frac{a}{\cos^2\theta} * \sin\theta\ \frac{dy}{d\theta} = \frac{b}{\cos^2\theta}\ \frac{dy}{dx} = \frac{b}{a}\frac{1}{\sin\theta}$ For the perpendicular on the cure at a point (x,y) of the curve we have

$$k = -\frac{1}{\frac{dy}{dx}} = -a\frac{\sin\theta}{b}\ k_x = \frac{1}{\sqrt{(1+k^2)}} = \frac{b}{\sqrt{(b^2+a^2\sin^2\theta)}}\quad k_y = k * k_x = -\frac{a\sin\theta}{\sqrt{(b^2+a^2\sin^2\theta)}}$$

The left hyperbola has an equation $x = \frac{a}{\cos\theta} - d$ and the right hyperbola $x = -\frac{a}{\cos\theta} + d$ where $d = \frac{a}{\cos(\theta_{max})}$ and the source is at $x_0 = -c - d$

For the case of a point at $(x_0\ 0)$ to find the incidence angle we have to find the cross product

$$\vec{k}X\vec{D} = k_x D_y - k_y D_x \text{ where } D_x = x - x_0 =$$

$$k_x D_y - k_y D_x = \frac{b}{\sqrt{(b^2+a^2\sin^2\theta)}}\, b\tan\theta + \frac{a\sin\theta}{\sqrt{(b^2+a^2\sin^2\theta)}}\left(\frac{a}{\cos\theta} + a\,n\right) =$$

$$\frac{an^2}{\sqrt{(n^2-\cos^2\theta)}}\tan\theta + \frac{\alpha\ \sin\theta}{\sqrt{(n^2-\cos^2\theta)}}\, n = \frac{a(n^2+n\cos\theta)}{\sqrt{(n^2-\cos^2\theta)}}\tan\theta$$

$$|D| = \sqrt{\left(\frac{a}{\cos\theta} + an\right)^2 + b^2\tan^2\theta} = \frac{1}{\cos\theta}\sqrt{(\alpha + an\cos\theta)^2 + (n^2-1)a^2\sin^2\theta} = \frac{a(n+\cos\theta)}{\cos\theta}$$

$$\alpha = \sin(incidence) = \frac{\frac{an(n+\cos\theta)}{\sqrt{(n^2-\cos^2\theta)}}\tan\theta}{\frac{a(n+\cos\theta)}{\cos\theta}} = \frac{n\sin\theta}{\sqrt{(n^2-\cos^2\theta)}}$$

$$\beta = \sin(refraction) = \frac{Sin(incidence)}{n} = \frac{\sin\theta}{\sqrt{(n^2-\cos^2\theta)}}$$

The angle φ of the perpendicular to the curve with the x-axis has tangent equal to $k = -a\frac{\sin\theta}{b}$ so

$sin\varphi = \frac{a\frac{sin\theta}{b}}{\frac{\sqrt{(b^2+a^2 sin^2\theta)}}{b}} = \frac{sin\theta}{\sqrt{(n^2-\cos^2\theta)}} = \beta$ so the inclination of the ray

has ω=0

In the case the source is at (x_0, y_0) then we need to find the intersection of the incident ray, with the left surface, the refracted ray with the right surface of the lens and the emerging ray. The incident ray is chosen to be to a point of the left hyperbola. To find the refracted ray we proceed as usual by finding the incidence angle and the refraction angle. In the case of Ibn Sahl's lens the derivation of the refracted ray proceeds as in the case of the spherical lens since the formula of the back lens is a simple function of the parameter and we do not need to find the point on the front surface as in the case of the Huygens' lens. Ibn Sahl's proof which utilizes geometrical theorems due to Apollonius are presented by Mihas [15].

Conclusion

The use of Excel files can help the students to connect the caustics with the movement of the wave fronts. The wave fronts touch the caustics and in many cases present a kink at the point of touch. This is seen in the caustic of the emerging rays for the rainbow.

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