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Exact Analytical Solutions for a Nonstationary Linear Inverse Problem of Heat Conduction for Bodies of One-Dimensional Geometry with Boundary Conditions on One Surface, as Well as on Two Surfaces for a Plane Body, a Cold Cylinder and a Hollow Sphere, Obtained in a Closed Recurrent Form

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Annotation
In this paper, exact analytical solutions for the non-stationary linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary non-stationary conditions on one and two boundary surfaces are obtained in a closed recurrent form. The recurrent form of writing the solution of the non-stationary linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary non-stationary temperature conditions on one and two boundary surfaces is a closed-form solution from unified positions, which is not always possible in an explicit form.

Keywords: Thermal Conductivity, Analytical, Non-stationary, Linear, One-dimensional, Flat, Spherical, Cylindrical, Inverse problem, Surface, Boundary conditions, Recurrent.

Introduction Relevance of the Application of Inverse Problems of Heat Conduction and Heat Transfer
Direct mathematical modeling makes it possible to predict the thermal state of a wide range of operating modes, for example, of a technical system, to analyze the influence of various factors on the behavior of this system, and to select optimal thermal modes. The use of direct methods of mathematical modeling requires an analysis of the accuracy of mathematical models. The model can have a very complex structure and take into account a fairly large number of factors. However, in this case, it is necessary to set the numerical values of all the characteristics included in the model, in particular, the thermophysical properties of materials, characteristics of thermal interaction with the washing medium, etc. If information is absent or has low accuracy, then the complex mathematical model loses its merits and does not provide the required forecast accuracy thermal modes. The practical application of mathematical modeling of heat transfer shows that the possible unsatisfactory accuracy in mathematical modeling, for example, of high-intensity thermal processes is due to the low accuracy of determining the characteristics using traditional direct methods [19]. In such cases, the use of computational and experimental methods, which are based on the principles of identifying systems with distributed parameters, based on algorithms and methods for solving various types of ill-posed inverse heat transfer problems [19], can be very effective. high-intensity thermal processes due to the low accuracy of determining the characteristics using traditional direct methods [19]. In such cases, the use of computational and experimental methods, which are based on the principles of identifying systems with distributed parameters, based on algorithms and methods for solving various types of ill-posed inverse heat transfer problems [19], can be very effective. high-intensity thermal processes due to the low accuracy of determining the characteristics using traditional direct methods [19]. In such cases, the use of computational and experimental methods, which are based on the principles of identifying systems with distributed parameters, based on algorithms and methods for solving various types of ill-posed inverse heat transfer problems [19], can be very effective.

As is known, in direct problems the required temperature is the temperature field, which is found as a solution of the heat equation with known internal transfer parameters corresponding to known boundary and initial conditions, and in inverse heat conduction problems the initial temperature distribution and boundary conditions are unknown functions to be determined. Inverse problems are divided into two main types: a) determination of the parameters of internal energy transfer - the coefficients of thermal and thermal diffusivity, heat capacity, light absorption coefficients, etc., which are the physical characteristics of a substance; b) determination of the conditions for external exchange of energy between the body and the environment, i.e. finding boundary conditions: this includes calculating the temperature of the outer surface and the heat flux passing through it, calculation of variable heat transfer coefficients, thermal contact resistances, degrees of emissivity, angular coefficients of irradiation, position of the surface of a phase transition or destruction, drawing up non-stationary balances of power and energy, etc. [nineteen]. It is clear that it is much more difficult to obtain a solution to the inverse problem of heat conduction than to the direct problem, however, in the direct problem, when measuring or implementing the given boundary conditions, many experimental obstacles can arise. Physical conditions are, for example, such that it is practically not always possible to install the sensor on the surface of the body or the measurement accuracy is significantly reduced due to the placement of the sensors. Consequently, it is often difficult to measure the law of change in the temperature of the heated surface of a solid. It is much easier to perform sufficiently accurate measurements of the temporal dependences of temperature at internal points on the thermally insulated surface of the body. Thus, the problem arises of choosing between relatively imprecise measurements and a complex analytical task. At the same time, a sufficiently accurate and easily realizable solution to the inverse problem would simultaneously reduce both difficulties to a minimum [19]. The direct problem of heat conductivity under correctly stated conditions has a unique solution. In the case of inverse problems, the identity of the temperature fields is possible as a result of external influences that are different in nature, but energetically equivalent [5, 6, 19]. The temperature field of a solid does not uniquely determine the boundary conditions under which it arose. A number of boundary conditions, which are energetically equivalent in their effects on the system, can reflect complex temperature processes in different ways. An example is the fact that any redistribution of heat flux densities, for example, between convective and radiation components, when combined, leads to an identical thermal state of the system [19]. There are also other disadvantages inherent in the reverse methods of studying unsteady heat transfer in technical systems: limitation of the number of points in the details, in which temperatures and heat fluxes are measured; experimentally determined values of temperatures and heat fluxes, on the basis of which the calculations are made, contain measurement errors even when using precision instruments, since placing the sensors in a solid to some extent violates the temperature field of the parts; surface curvature, spatial and temporal changes in heat fluxes in the body do not make it possible to accurately predict the direction of the heat flux, or in other words, to determine the location of the sensor, which should be on the normal to the surface. It should be noted that inverse methods do not provide the possibility of physical interpretation of nonstationary complex processes occurring in systems. In addition to the disadvantages, including the above, the reverse methods have some advantages over the direct ones. In the direct problem, when measuring or realizing the given boundary conditions, many experimental obstacles can arise. The physical conditions in the systems under study can be as follows: that it is impossible to install the sensor on the surface of the body (for example, on the surface of coatings) or the measurement accuracy is significantly reduced due to the placement of the sensors, therefore it is often difficult to measure the law of change in temperatures and heat fluxes of surfaces of solids. Summarizing the above, we can conclude that there is a relevance of obtaining in a unified form an exact closed analytical solution of the nonstationary linear inverse problem of heat conduction for bodies of one-dimensional geometry with boundary conditions on one or two surfaces. In research, the exact closed analytical solution of this inverse problem of heat conduction is achieved in a recurrent form, i.e. in an implicit form, since this is not always possible in an explicit form [2-6]. on the surface of coatings) or the measurement accuracy is significantly reduced due to the placement of sensors, therefore it is often difficult to measure the law of change in temperatures and heat fluxes of surfaces of solids. Summarizing the above, we can conclude that there is a relevance of obtaining in a unified form an exact closed analytical solution of the nonstationary linear inverse problem of heat conduction for bodies of one-dimensional geometry with boundary conditions on one or two surfaces. 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Solutions in a Recurrent Form for a Nonstationary Linear Inverse Problem of Heat Conduction for Bodies of One-Dimensional Geometry With Boundary Conditions on one Surface

Existing exact solutions of the inverse problems of unsteady heat conduction are relatively few in number, and there are significantly fewer of them than the corresponding solutions of the direct problem of unsteady heat conduction. It can be pointed out that one of the first successful attempts to solve the inverse problem of unsteady heat conduction for a flat body was first undertaken in 1890 by J. Stephan [1]. Subsequently, for a one-dimensional linear inverse non-stationary heat conduction problem, solutions were obtained in an independent way by OR Burggraf and D. Langford under the assumption that the non-stationary heat flux density and temperature are known at the point of location of the sensor [2, 3]. The exact solutions for the temperature fields from the previously known temperatures at two different internal points by the integral Laplace transform were obtained by M. Imber and D. Khan [4]. Similar solutions for one-dimensional bodies are also given in [5, 6] in which solutions for unsteady temperatures are given explicitly, and the heat flux density was determined by differentiating the temperature fields. Subsequently, solutions were obtained to similar problems, partly having not only theoretical but also applied character, including the nonlinear one-dimensional problem of nonstationary heat conduction [7–19]. As was partly indicated in [2-6], the explicit expression of solutions for a nonstationary linear inverse problem of heat conduction for bodies of one-dimensional geometry is not possible in all cases; therefore, in order to obtain the final solution, additional assumptions have to be applied, for example, as in [2] where the thin-wall assumption is used. The aim of the study is to obtain a solution to the nonstationary linear inverse problem of heat conduction for bodies of one-dimensional geometry with boundary conditions on one surface from single positions in a closed recurrent form, which will have certain advantages over solutions in an explicit form, since they can be obtained for all the above problems, and explicitly - not for everyone. Let us write the equation of nonlinear unsteady heat conductivity for a body of one-dimensional geometry and constant curvature (in this case, the radial coordinate is considered) in the following form [5] since they can be obtained for all of the above tasks, but explicitly - not for all. Let us write the equation of nonlinear unsteady heat conductivity for a body of one-dimensional geometry and constant curvature (in this case, the radial coordinate is considered) in the following

$$(1)\ \frac{1}{a}\frac{\partial t}{\partial \tau} = \nabla^2 t = r^{1-k}\frac{\partial}{\partial r}\left(r^{k-1}\frac{\partial t}{\partial r}\right) = \frac{\partial^2 t}{\partial r^2} + \frac{k-1}{r}\frac{\partial t}{\partial r}$$

where k is the number of final measurements: 1 - flat field; 2 - cylindrical; 3 - spherical; t is the temperature; r - radial coordinate; a - coefficient of thermal diffusivity. The domain of definition of the differential equation (1) is from 0 to r2 (radial coordinate of the outer surface) along the coordinate (in the case of hollow bodies: from r1 (radial coordinate of the inner surface) to r2 and from 0 to the current valueτ by time (τ> 0). In dimensionless form, this equation can be written as follows [5].

$$(2)\ \frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial^2 T}{\partial \rho^2} + \frac{k-1}{\rho}\frac{\partial T}{\partial \rho}$$

where is the Fourier criterion; T is the dimensionless temperature; ρ = r / r1 - dimensionless coordinate; r1 is the radial coordinate at which the boundary conditions are set. The inverse problem of heat conduction for Eqs. (1) or (2) consists in finding the boundary conditions on the surface of a one-dimensional body at known non-stationary temperature and heat flux and thermophysical characteristics of the body material, independent of temperature. The study investigates the process of heat conduction at a time that is sufficiently distant from the initial time, so the influence of the initial conditions has practically no effect on the temperature distribution at the time of measurement or observation (the so-called "problem without initial conditions"). In practical terms, this may mean that at a sufficient distance from the initial time of the aftereffect component, taking into account the influence of the initial conditions, becomes so small that it will already be less than the measurement error of sensors that measure temperatures and heat fluxes [5, 6]. The component of the effect of the temperature field of a one-dimensional layer, which is heated on the inner surface, is considered using a dimensionless coordinate for which the heated surface corresponds to a unit value (the homochronism complex refers to a given internal radial coordinate), can be represented in the following form [5].

$$\mathrm{Fo} = \frac{a\tau}{r_1^2}$$

$$T(\rho,\mathrm{Fo}) = \sum_{n=0}^{\infty} T_1^{(n)}(\mathrm{Fo})\,P_{n,1} + \sum_{n=0}^{\infty} \mathrm{Ki}^{(n)}(\mathrm{Fo}) = \sum_{n=0}^{\infty}\Theta_{n,1}\,P_{n,1} + \sum_{n=0}^{\infty}\Theta_{n,2}\,P_{n,2},$$

where is the Kirpichev criterion; - Fourier criterion; ρ = r / r1 - dimensionless coordinate; r1 —radial coordinate at which the boundary conditions are set; a - coefficient of thermal diffusivity; λ is the thermal conductivity coefficient; q is the heat flux density; Δt is the temperature difference. On the heated surface, the boundary condition of the second kind takes place. In this case, the heat flux density and temperature are measured on the same surface. Solutions for bodies of simple configuration will differ in the values of the radial quasi-polynomials Pn, 1 and Pn, 2. Within the framework of this work, these quasi-polynomials will be solved in recurrent forms, in contrast to [2-6] and [7-19].

$$\mathrm{Ki} = \frac{q r_1}{\lambda \Delta t}\,\mathrm{Fo} = \frac{a\tau}{r_1^2}$$

Flat Plate

Quasi-polynomials Pn, 1 and Pn, 2 for a flat plate will be as follows:

$$P_{n+1,1} = \int_0^\rho \int_0^\rho P_{n,1}\,d\rho d\rho;\ (4)$$

$$P_{n+1,2} = \int_0^\rho \int_0^\rho P_{n,2}\,d\rho d\rho;\ (\text{five})$$

$$P_{0,1} = 1;\ P_{0,2} = \rho.\ (6)$$

For the first quasi-polynomials P1,1 and P2,1, etc., P1,2 and P2,2, etc. for a flat plate, you can write:

$$P_{1,1} = \frac{\rho^2}{2};\ P_{2,1} = \frac{\rho^4}{24};\ P_{3,1} = \frac{\rho^6}{720};\ \dots;\ \dots;\ (7)$$

$$P_{1,2} = \frac{\rho^3}{6};\ P_{2,2} = \frac{\rho^5}{120};\ P_{3,2} = \frac{\rho^7}{5040};\ \dots;\ \dots;\ .\ (8)$$

Therefore, using the method of mathematical induction, one can writeFind quasi-polynomials for solving the inverse non-stationary heat conduction problem when setting the boundary conditions on the same surface for a flat plate in recurrent form:

$$P_{n,1} = \frac{\rho^2}{2n\cdot(2n-1)}P_{n-1,1};\ (9)$$

$$P_{n,2} = \frac{\rho^2}{2n\cdot(2n+1)}P_{n-1,2}.\ (10)$$

Solid Cylinder

Quasi-polynomials Pn, 1 for a solid cylinder will be as follows:

$$P_{n+1,1} = \int_0^\rho \frac{1}{\rho}\int_0^\rho \rho P_{n,1}\,d\rho d\rho;\ (11)$$

$$P_{0,1} = 1.\ (12)$$

$$P_{1,1} = \frac{\rho^2}{4};\ P_{2,1} = \frac{\rho^4}{64};\ P_{3,1} = \frac{\rho^4}{2304};\ \dots;\ \dots;\ .\ (13)$$

Therefore, using the method of mathematical induction, one can writeFind quasi-polynomials for solving the inverse non-stationary heat conduction problem when setting the boundary condition on the axis of a solid cylinder in a recurrent form:

$$P_{n,1} = \frac{\rho^2}{4n^2}P_{n-1,1}.\ (14)$$

Hollow Cylinder

Quasi-polynomials Pn, 1 and Pn, 2 for a hollow cylinder will be as follows:

$$P_{n+1,1} = \int_1^\rho \frac{1}{\rho}\int_1^\rho \rho P_{n,1}\,d\rho d\rho;\ (15)$$

$$P_{n+1,2} = \int_1^\rho \frac{1}{\rho}\int_1^\rho \rho P_{n,2}\,d\rho d\rho;\ (16)$$

$$P_{0,1} = 1;\ P_{0,2} = \ln\rho.\ (17)$$

For the first quasi-polynomials P1,1 and P2,1, P1,2 and P2,2, etc. for a hollow cylinder, you can write:

$$P_{1,1} = \frac{1}{4}\rho^2 - \frac{1}{2}\ln\rho - \frac{1}{4} = \frac{1}{4}\rho^2 - \frac{1}{2}P_{0,2} - \frac{1}{4}P_{0,1}; (18)$$

$$P_{1,2} = \frac{1}{4}\rho^2\ln\rho - \frac{1}{4}\rho^2 + \frac{1}{4}\ln\rho + \frac{1}{4} = \frac{1}{4}\rho^2(\ln\rho - 1) + \frac{1}{4}P_{0,2} + \frac{1}{4}P_{0,1}; \text{(nineteen)}$$

$$P_{2,1} = \frac{1}{64}\rho^4 - \frac{1}{8}\rho^2\ln\rho + \frac{1}{16}\rho^2 - \frac{1}{16}\ln\rho - \frac{5}{64} =$$

$$(20) = \frac{1}{64}\rho^4 - \frac{1}{2}P_{1,2} - \frac{1}{4}P_{1,1} - \frac{1}{16}P_{0,2} - \frac{1}{64}P_{0,1};$$

$$P_{2,2} = \frac{1}{64}\rho^4\ln\rho - \frac{3}{128}\rho^4 + \frac{1}{16}\rho^2\ln\rho + \frac{1}{64}\ln\rho + \frac{3}{128} =$$

$$(21) = \frac{1}{64}\rho^4\left(\ln\rho - \frac{3}{2}\right) + \frac{1}{4}P_{1,1} + \frac{3}{128}P_{0,1} + \frac{1}{4}P_{1,2} + \frac{5}{64}P_{0,2};$$

$$P_{3,1} = \frac{1}{2304}\rho^6 - \frac{1}{128}\rho^4\ln\rho + \frac{1}{128}\rho^4 - \frac{1}{64}\rho^2\ln\rho - \frac{1}{256}\rho^2 - \frac{1}{384}\ln\rho - \frac{5}{1152} =$$

$$; \ldots; \ldots; (22) = \frac{1}{2304}\rho^6 - \frac{1}{4}P_{2,1} - \frac{1}{64}P_{1,1} - \frac{1}{2304}P_{0,1} - \frac{1}{2}P_{2,2} - \frac{1}{16}P_{1,2} - \frac{1}{384}P_{0,2}$$

$$P_{3,2} = \frac{1}{2304}\rho^6\ln\rho - \frac{11}{13824}\rho^6 + \frac{1}{256}\rho^4\ln\rho - \frac{1}{512}\rho^4 + \frac{1}{256}\rho^2\ln\rho + \frac{1}{512}\rho^2 + \qquad (23)$$

$$+\frac{1}{2304}\ln\rho + \frac{11}{13824} =$$

$$= \frac{1}{2304}\rho^6\left(\ln\rho - \frac{11}{6}\right) + \frac{1}{4}P_{2,1} + \frac{3}{128}P_{1,1} + \frac{11}{13824}P_{0,1} + \frac{1}{4}P_{2,2} + \frac{5}{64}P_{1,2} + \frac{5}{1152}P_{0,2}; \ldots;.$$

Therefore, using the method of mathematical induction, one can write down quasi-polynomials for solving the inverse non-stationary heat conduction problem when setting the boundary condition on the inner surface of a hollow cylinder in recurrent form:

$$P_{n,1} = \frac{1}{((2n)!!)^2}\rho^2 - \sum_{m=0}^{n-1}\frac{1}{\left((2(n-m))!!\right)^2}P_{m,1} - \sum_{m=0}^{n-1}\frac{2(n-m)}{\left((2(n-m))!!\right)^2}P_{m,2}; \; (24)$$

$$P_{n,2} = \left(\ln\rho - \sum_{m=1}^{n} m^{-1}\right)\rho^2 + \sum_{m=0}^{n-1}\frac{1}{\left((2(n-m))!!\right)^2}\sum_{l=1}^{n-m} l^{-1}P_{m,1} +$$

$$+\sum_{m=0}^{n-1}\frac{1}{\left((2(n-m))!!\right)^2}P_{m,2} + \sum_{m=0}^{n-1}\frac{2(n-m)}{\left((2(n-m))!!\right)^2}\sum_{l=1}^{n-m-1} l^{-1}P_{m,2}. \; (25)$$

Solid Ball

Quasi-polynomials Pn, 1 for a solid ball will be as follows:

$$P_{n+1,1} = \int_0^\rho \frac{1}{\rho^2}\int_0^\rho \rho^2 P_{n,1}\, d\rho d\rho; \; (26)$$

$$P_{0,1} = 1\,. \; (27)$$

For the first quasi-polynomials Pn, 1, etc. for a solid ball, you can write:

$$P_{1,1} = \frac{\rho^2}{6}; P_{2,1} = \frac{\rho^4}{120}; \; P_{3,1} = \frac{\rho^4}{5040}; \ldots; \ldots;. \; (28)$$

Therefore, using the method of mathematical induction, one can writeFind quasi-polynomials for solving the inverse non-stationary problem of heat conduction when setting the boundary condition at the center of a solid ball in a recurrent form:

$$P_{n,1} = \frac{\rho^2}{2n\cdot(2n+1)}P_{n-1,1} \cdots \; (29)$$

Hollow Ball

Quasi-polynomials Pn, 1 and Pn, 2 for a hollow ball will be as follows:

$$P_{n+1,1} = \int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2 P_{n,1}\, d\rho d\rho; \; (30)$$

$$P_{n+1,2} = \int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2 P_{n,2}\, d\rho d\rho; \; (31)$$

$$P_{0,1} = 1; \; P_{0,2} = 1 - \frac{1}{\rho}. \; (32)$$

For the first quasi-polynomials P1,1 and P2,1, P1,2 and P2,2, etc. for a hollow ball, you can write:

$$P_{1,1} = \frac{1}{6}\frac{1}{\rho}(\rho-1)^2(\rho+2); \; P_{2,1} = \frac{1}{120}\frac{1}{\rho}(\rho-1)^4(\rho+4); \; (33)$$

$$P_{3,1} = \frac{1}{5040}\frac{1}{\rho}(\rho-1)^6(\rho+6); \ldots; \ldots;. \; (34)$$

$$P_{1,2} = \frac{1}{6}\frac{1}{\rho}(\rho-1)^3; P_{2,2} = \frac{1}{120}\frac{1}{\rho}(\rho-1)^5; P_{3,2} =$$

$$\frac{1}{5040}\frac{1}{\rho}(\rho-1)^7; \ldots; \ldots;. \; (35)$$

Therefore, using the method of mathematical induction, one can write down quasi-polynomials for solving the inverse non-stationary heat conduction problem when setting the boundary condition on the inner surface of the hollow sphere in recurrent form:

$$P_{n,1} = \frac{1}{2n\cdot(2n+1)}(\rho-1)^2\frac{(\rho+2n)}{(\rho+2(n-1))}P_{n-1,1}\ldots \quad (36)$$

$$P_{n,2} = \frac{1}{2n\cdot(2n+1)}(\rho-1)^2 P_{n-1,2}\ldots \quad (37)$$

For given unsteady boundary conditions on one surface Θn, 1 and Θn, 2 the re-currence relations are as follows:

$$\Theta_{n,i} = \frac{r_1^2}{a}\frac{\partial\Theta_{n-1,i}}{\partial\tau}, \forall\, i = 1,2\ldots \quad (38)$$

The above relations express the recurrent form of the exact solution to the inverse problem of unsteady heat conductionfor bodies of one-dimensional geometry with non-stationary boundary conditions specified on one side. The recurrent form of writing the solution makes it possible to solve this problem from a unified position in a closed form, since the expression of solutions in explicit form, as, for example, in [7-19], is possible not in all cases, as indicated in [2, 5, 6] The issues of correctness of this inverse problem of heat conduction (i.e, the existence of a solution, its uniqueness and its stability) were considered in detail in [5, 6], therefore, in this study there is no need to reconsider them. The above solutions of the non-stationary inverse problem of heat conduction for one-dimensional bodies, obtained in this work, were successfully applied in a practical way as an integral part of the conjugate problem in determining the maximum effect of the carbon layer on the surface of the combustion chamber on the non-stationary parameters of the working fluid during radiation-convective heat transfer [20-22], and also in the development of the theory of heat transfer in heat-insulating packaging to stabilize the temperature regimes of storage of perishable products [23-24]. For heat transfer conditions typical of [23–24], calculations were carried out using the dependences generated in this study. With the same temperature boundary condition, the largest deviation will be for a flat body, and the smallest for a solid sphere; for a solid cylinder, there will be an intermediate value. For both a hollow cylinder and a hollow ball, the temperature deviation will be larger than for a solid cylinder and a ball, respectively. Comparison of a hollow cylinder with a hollow ball shows that for small values of r2 / r1, the deviation for a hollow cylinder will be less than for a hollow ball, but for large values of r2 / r1, the deviation for a hollow cylinder will already be greater than for a hollow ball. For the conditions considered in [23-24], the above break occurs at a value of r2 / r1 ≈ 3 2/15. The analysis of the calculations performed indicates a stronger dependence of the calculated temperature on the parameter r2 / r1 for a hollow sphere than for a hollow cylinder. Comparison of a hollow cylinder with a hollow ball shows that for small values of r2 / r1, the deviation for a hollow cylinder will be less than for a hollow ball, but for large values of r2 / r1, the deviation for a hollow cylinder will already be greater than for a hollow ball. For the conditions considered in [23-24], the above break occurs at a value of r2 / r1 ≈ 3 2/15. The analysis of the calculations performed indicates a stronger dependence of the calculated temperature on the parameter r2 / r1 for a hollow sphere than for a hollow cylinder. Comparison of a hollow cylinder with a hollow ball shows that for small values of r2 / r1, the deviation for a hollow cylinder will be less than for a hollow ball, but for large values of r2 / r1, the deviation for a hollow cylinder will already be greater than for a hollow ball. For the conditions considered in [23-24], the above break occurs at a value of r2 / r1 ≈ 3 2/15. The analysis of the calculations performed indicates a stronger dependence of the calculated temperature on the parameter r2 / r1 for a hollow sphere than for a hollow cylinder.

Solutions In a Recurrent Form for a Nonstationary Linear Inverse Problem of Heat Conduction for Bodies of One-Dimensional Geometry with Boundary Temperature Conditions on Both Surfaces

Temperature fields of hollow cylinders and spheres, plates, whose faces are in different media, are asymmetric, but one-dimensional. An asymmetric temperature field is obtained from measurements of temperatures at the boundaries of a body, which should be known in advance as functions of time. The component of the effect of the temperature field of a one-dimensional layer, on the boundaries of which there are non-stationary temperature boundaries, is considered when using a dimensionless coordinate: the first point is taken as the origin of coordinates, and the second has a unit abscissa (for a flat field); the first point has a unit abscissa, and the second has a point ρ2 (for spherical and cylindrical fields) can be represented in the following form [5].

$$T(\rho,\mathrm{Fo}) = \sum_{n=0}^{\infty} T_1^{(n)}(\mathrm{Fo})\,P_{n,1} + \sum_{n=0}^{\infty} T_2^{(n)}(\mathrm{Fo})\,P_{n,2} =$$

$$= \sum_{n=0}^{\infty}\Theta_{n,1}\,P_{n,1} + \sum_{n=0}^{\infty}\Theta_{n,2}\,P_{n,2}\,. \quad (39)$$

On both surfaces, the boundary condition of the first kind holds. In this case, temperatures are measured at the boundary surfaces. Solutions for bodies of simple configuration will differ in the values of the radial quasi-polynomials Pn, 1 and Pn, 2. Within the framework of this work, these quasi-polynomials will be solved in recurrent forms, in contrast to [2-6] and [7-19].

Recursive Form Solutions for a Nonstationary Linear Inverse Heat Conduction Problem for Flat Plates and Hollow Spheres (Bodies of One-Dimensional Geometry) With Boundary Temperature Conditions on Both Surfaces without Using Bernoulli Numbers

In this section, the problem is posed to obtain the data of recurrent solutions without using the Bernoulli numbers Bn, but using the solution method based on recurrent dependencies, which are similarly used for plane, cylindrical and spherical fields.

Flat Plate (Solution Without using Bernoulli Numbers Bn)

Quasipolynomials for solving the inverse unsteady heat conduction problem when setting the tem-perature boundary conditions on both boundary surfaces for a flat plate in a recurrent form are ob-tained from the solutions obtained for the inverse nonstationary heat conduction problem when set-ting the boundary condition on the "inner" surface of the flat plate, i.e. formulas (9) and (10). Quasi-polynomials Pn, 1 and Pn, 2 for a flat plate will be as follows:

$$P_{n+1,1} = \int_0^\rho\int_0^\rho P_{n,1}\,d\rho d\rho - \rho\int_0^1\int_0^\rho P_{n,1}\,d\rho d\rho; \quad (40)$$

$$P_{n+1,2} = \int_0^\rho\int_0^\rho P_{n,2}\,d\rho d\rho - \rho\int_0^1\int_0^\rho P_{n,2}\,d\rho d\rho; \quad (41)$$

$$P_{0,1} = 1-\rho;\, P_{0,2} = \rho. \quad (42)$$

For the first quasi-polynomials P1,1 and P2,1, P1,2 and P2,2, etc. for a flat plate, you can write:

$P_{1,1} = -\frac{1}{6}\rho^3 + \frac{1}{2}\rho^2 - \frac{1}{3}\rho$; (43)

$P_{2,1} = -\frac{1}{120}\rho^5 + \frac{1}{24}\rho^4 - \frac{1}{18}\rho^3 + \frac{1}{45}\rho$; (44)

$P_{3,1} = -\frac{1}{5040}\rho^7 + \frac{1}{720}\rho^6 - \frac{1}{360}\rho^5 + \frac{1}{270}\rho^3 - \frac{2}{945}\rho$; (45)

$P_{4,1} = -\frac{1}{362880}\rho^9 + \frac{1}{40320}\rho^8 - \frac{1}{15120}\rho^7 + \frac{1}{5400}\rho^5 - \frac{1}{2835}\rho^3 + \frac{1}{4725}\rho; \ldots; \ldots;$. (46)

$P_{1,2} = \frac{1}{6}\rho^3 - \frac{1}{6}\rho$; (47)

$P_{2,2} = \frac{1}{120}\rho^5 - \frac{1}{36}\rho^3 + \frac{7}{360}\rho$; (48)

$P_{3,2} = \frac{1}{5040}\rho^7 - \frac{1}{720}\rho^5 + \frac{7}{2160}\rho^3 - \frac{31}{15120}\rho$; (49)

$P_{4,2} = \frac{1}{362880}\rho^9 - \frac{1}{30240}\rho^7 + \frac{31}{43200}\rho^5 - \frac{31}{90720}\rho^3 + \frac{127}{604800}\rho; \ldots ; \ldots;$. (50)

Here it is convenient to introduce local notation, which is valid only for this section, in order to avoid discrepancies in the solution of the problem in the future:

$F_{n,1} \overset{\text{def}}{=} P_{n,1}\big|^{(9)}$; (51)

$F_{n,2} \overset{\text{def}}{=} P_{n,2}\big|^{(10)}$. (52)

In other words, the functions Fn, 1 and Fn, 2 in the framework of this section denote the quasi-polynomials Pn, 1 and Pn, 2 for the inverse nonstationary heat conduction problem when the boundary condition on the inner surface of a flat plate is specified from formulas (9) and (10), respectively. First we solve the problem for Pn, 2, since it is simpler than for Pn, 1; the solution of the first problem will be taken as a basis for solving the second problem. We rewrite the anti-Laplacians P1,2 in the following form:

$P_{1,2} = \int_0^\rho \int_0^\rho \rho \, d\rho d\rho - \int_0^1 \int_0^\rho \rho \, d\rho d\rho = F_{1,2} - F_{0,2}F_{1,2}\big|_{\rho=1}$; (53)

In order to obtain a recurrent solution to this problem, we rewrite the last expression as follows:

$P_{1,2} = F_{1,2} - F_{0,2}\Phi_{1,2}\big|_{\rho=1}$, (54)

Where. Subsequent anti-Laplacians for quasi polynomials Pn, 2 will be as follows: $\Phi_{1,2}=F_{1,2}$

$P_{2,2} = F_{2,2} - F_{1,2}\Phi_{1,2}\big|_{\rho=1} - F_{0,2}\Phi_{2,2}\big|_{\rho=1}$, (55)

Where; $\Phi_{2,2} = F_{2,2} - F_{1,2}\Phi_{1,2}\big|_{\rho=1}$

$P_{3,2} = F_{3,2} - F_{2,2}\Phi_{1,2}\big|_{\rho=1} - F_{1,2}\Phi_{2,2}\big|_{\rho=1} - F_{0,2}\Phi_{3,2}\big|_{\rho=1}$, (56)

Where; $\Phi_{3,2} = F_{3,2} - F_{2,2}\Phi_{1,2}\big|_{\rho=1} - F_{1,2}\Phi_{2,2}\big|_{\rho=1}$

$P_{4,2} = F_{4,2} - F_{3,2}\Phi_{1,2}\big|_{\rho=1} - F_{2,2}\Phi_{2,2}\big|_{\rho=1} - F_{1,2}\Phi_{3,2}\big|_{\rho=1} - F_{0,2}\Phi_{4,2}\big|_{\rho=1}$, ...; ...; (57)

Where. $\Phi_{4,2} = F_{4,2} - F_{3,2}\Phi_{1,2}\big|_{\rho=1} - F_{2,2}\Phi_{2,2}\big|_{\rho=1} - F_{1,2}\Phi_{3,2}\big|_{\rho=1}$

Hence, n-th degree anti-Laplacians for quasi-polynomials Pn, 2 can be written in the following form:

$$P_{n,2} = F_{n,2} - \sum_{i=0}^{n-1} F_{n-1-i,2}\Phi_{i+1,2}\big|_{\rho=1}. \quad (58)$$

As can be seen from the formula (58), for its solution it is used as a "direct" recurrence, i.e. the use of the previous members of the series for the current member of a series, and "partial" recurrence, i.e. use in the output for the current member of a part of the same member of the series. Now the func-tion Φi, 2 should be determined. For this, a form should be formalized for them. Let us rewrite Eq. (54) for Φ1,2 in the form typical for larger values of the parameter i, namely:

$$\Phi_{1,2} = F_{1,2} - F_{0,2}\Phi_{0,2}\big|_{\rho=1} \ldots (59)$$

For the expression Φ1,2 from formula (59) to be identically equal to its definition from (54), it is necessary (since F0,2= ln ρ) so that:

$$\Phi_{0,2}\big|_{\rho=1} = 0 \; (60)$$

After the last formalization, we can write a closed expression for Φi, 2:

$$\Phi_{i,2} = F_{i,2} - \sum_{k=0}^{i-1} F_{i-k,2}\Phi_{k,2}\big|_{\rho=1}. (61)$$

Thus, expressions (58), (61), (60) give an exact solution to the quasi-polynomial problem for solving the inverse nonstationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a flat plate in a recurrent form. As can be seen, the solution for the functions Φi, 2 formally contains terms with. Obviously, all these terms are absent, for example, in (54) - (58). This is quite natural, since in the accepted representation (59) for Φi, 2 these terms are fictitious and equal (since) to zero: In the solution for Pn, 2 (58) there are no terms with Φ0.2, but there are terms with F0.2. The solution for Φi, 2 (61) lacks terms with F0,2, but formally there are terms with identically zero.

$$F_{i,2}\Phi_{0,2}\big|_{\rho=1}.\ \Phi_{0,2}\big|_{\rho=1} = 0 F_{i,2}\Phi_{0,2}\big|_{\rho=1} \equiv 0.\ F_{i,2}\Phi_{0,2}\big|_{\rho=1}$$

Now you should get a solution for quasi-polynomials Pn, 1, using the above solution method and based on the already available solutions for Pn, 2, Fn, 1, Fn, 2. Addition (40) and (41) for the first terms gives the following expression:

$$P_{1,1} + P_{1,2} = \int_0^\rho\int_0^\rho d\rho d\rho - \rho\int_0^1\int_0^\rho d\rho d\rho =$$
$$= F_{1,1} - F_{0,2}F_{1,1}\big|_{\rho=1} = F_{1,1} - F_{0,2}\Phi_{1,1}\big|_{\rho=1}, (62)$$

Where. In what follows, for Pn, 1, we will proceed in the same way as in the solution for quasi-polynomials Pn, 2, namely: $\Phi_{1,1}=F_{1,1}$

$$P_{1,1} = F_{1,1} - F_{0,2}\Phi_{1,1}\big|_{\rho=1} - P_{1,2}\ , (63)$$

Where. Subsequent anti-Laplacians for quasi-polynomials Pn, 1 will be as follows: $\Phi_{1,1}=F_{1,1}$

$$P_{2,1} = F_{2,1} - F_{1,2}\Phi_{1,1}\big|_{\rho=1} - F_{0,2}\Phi_{2,1}\big|_{\rho=1} - P_{2,2}\ , (64)$$

where; $\Phi_{2,1} = F_{2,1} - F_{1,2}\Phi_{1,1}\big|_{\rho=1}$

$$P_{3,1} = F_{3,1} - F_{2,2}\Phi_{1,1}\big|_{\rho=1} - F_{1,2}\Phi_{2,1}\big|_{\rho=1} - F_{0,2}\Phi_{3,1}\big|_{\rho=1} - P_{3,2}, (65)$$

where; $\Phi_{3,1} = F_{3,1} - F_{2,2}\Phi_{1,1}\big|_{\rho=1} - F_{1,2}\Phi_{2,1}\big|_{\rho=1}$

$$P_{4,1} = F_{4,1} - F_{3,2}\Phi_{1,1}\big|_{\rho=1} - F_{2,2}\Phi_{2,1}\big|_{\rho=1} - F_{1,2}\Phi_{3,1}\big|_{\rho=1} -$$
$$-F_{0,2}\Phi_{4,1}\big|_{\rho=1} - P_{4,2}\ , \ldots, \ldots, (66)$$

Where. Therefore, n-th degree anti-Laplacians for quasi-polynomials Pn, 1 can be written in the

$$\Phi_{4,1} = F_{4,1} - F_{3,2}\Phi_{1,1}\big|_{\rho=1} - F_{2,2}\Phi_{2,1}\big|_{\rho=1} - F_{1,2}\Phi_{3,1}\big|_{\rho=1}$$

$$P_{n,1} = F_{n,1} - \sum_{i=0}^{n-1} F_{n-1-i,2}\Phi_{i+1,1}\big|_{\rho=1} - P_{n,2}. (67)$$

Now the function Φi, 1 should be determined. For this, a form should be formalized for them. Let us rewrite Eq. (67) for Φ1,1 in the form typical for larger values of the parameter i, namely:

$$\Phi_{1,1} = F_{1,1} - F_{0,2}\Phi_{0,1}\big|_{\rho=1} \ldots (68)$$

For the expression Φ1,1 from formula (68) to be identically equal to its definition from (62), it is necessary (since F0,2= ln ρ) so that:

$$\ldots (69)\Phi_{0,1}\big|_{\rho=1} = 0$$

After the last formalization, we can write a closed expression for Φi, 1:

$$\Phi_{i,1} = F_{i,1} - \sum_{k=0}^{i-1} F_{i-k,2}\Phi_{k,1}\big|_{\rho=1}. (70)$$

So the expressions (67), (70), (69) give an exact solution to the quasi-polynomial problem for solving the inverse non-stationary heat conduction problem when setting the temperature boundary condi-tions on both boundary surfaces for a flat plate in a recurrent form. As can be seen, the solution for the functions Φi, 1 formally contains terms with. Obviously, all these terms are absent, for example, in (62) - (70). This is quite natural, since in the accepted representation (70) for Φi, 1 these terms are fictitious and equal (since) to zero:

$$F_{i,1}\Phi_{0,1}\big|_{\rho=1}.\ \Phi_{0,1}\big|_{\rho=1} = 0 F_{i,2}\Phi_{0,1}\big|_{\rho=1} \equiv 0.$$

The solution for Pn, 1 (67) does not contain terms with Φ0.1, but there are terms with F0.2. The solution for Φi, 1 (70) lacks terms with F0,2, but formally there are terms with identically equal to zero. $F_{i,2}\Phi_{0,1}\big|_{\rho=1}$

In principle, the problem of the exact solution of quasi-polynomials Pn, 2 - (58), (61), (60) - and Pn, 1 - (67), (70), (69) - for the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a flat plate in a recurrent form can be completed. However, it is possible to write these solutions in a combined form, for which it is necessary to rewrite in the appropriate form the series for Φi, 1 and Φi, 2 from formulas (61) and (70), respectively:

$$\Phi_{i+1,1} = F_{i+1,1} - \sum_{k=0}^{i} F_{i+1-k,2}\Phi_{k,1}\big|_{\rho=1}. (71)$$

$$\Phi_{i+1,2} = F_{i+1,2} - \sum_{k=0}^{i} F_{i+1-k,2}\Phi_{k,2}\big|_{\rho=1}. (72)$$

In the combined form, the exact solutions of this problem (for Pn, 1 - (67) and for Pn, 2 - (58)) will look like this:

$$P_{n,1} = F_{n,1} - \sum_{i=0}^{n-1} F_{n-1-i,2}\left[F_{i+1,1} - \sum_{k=0}^{i} F_{i+1-k,2}\Phi_{k,1}\big|_{\rho=1}\right]\Bigg|_{\rho=1} - P_{n,2}. (73)$$

$$P_{n,2} = F_{n,2} - \sum_{i=0}^{n-1} F_{n-1-i,2}\left[F_{i+1,2} - \sum_{k=0}^{i} F_{i+1-k,2}\Phi_{k,2}\big|_{\rho=1}\right]\Bigg|_{\rho=1}. (74)$$

Hollow cylinder

Quasi-polynomials Pn, 1 and Pn, 2 for a hollow cylinder will be as follows:

$$P_{n+1,1} = \int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,1}\, d\rho d\rho - \frac{\ln\rho}{\ln\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,1}\, d\rho d\rho; \; (75)$$

$$P_{n+1,2} = \int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,2}\, d\rho d\rho - \frac{\ln\rho}{\ln\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,2}\, d\rho d\rho; \; (76)$$

$$P_{0,1} = 1 - \frac{\ln\rho}{\ln\rho_2}; P_{0,2} = \frac{\ln\rho}{\ln\rho_2}. \; (77)$$

For the first quasi-polynomials P1,1 and P2,1, P1,2 and P2,2, etc. for a hollow cylinder, you can write:

$$P_{1,1} = \frac{1}{4(\ln\rho_2)^2} \ln\rho\,[2(\ln\rho_2)^2 - \rho_2^2 + 2\ln\rho_2 + 1] -$$

$$- \frac{1}{4\ln\rho_2}[\ln\rho_2 + \ln\rho - \rho^2\ln\rho_2 + \rho^2\ln\rho - \rho^2 + 2\ln\rho_2\ln\rho + 1] \; (78)$$

$$P_{2,1} = \frac{1}{128(\ln\rho_2)^2}[13\ln\rho_2 + 8\ln\rho - 8\rho_2^2\ln\rho - 16\rho^2\ln\rho_2 + 3\rho^4\ln\rho_2 + \; +8\rho^2\ln\rho +$$

$$8(\ln\rho_2)^2\ln\rho - 8\rho_2^2 - 8\rho^2 + 6(\ln\rho_2)^2 + 8\rho_2^2\rho^2 - \; -8\rho^2(\ln\rho_2)^2 + +2\rho^4(\ln\rho_2)^2 +$$

$$14\ln\rho_2\ln\rho - 8\rho_2^2\rho^2\ln\rho + 8\rho^2\ln\rho_2\ln\rho - \; -2\rho^4\ln\rho_2\ln\rho + 8] -$$

$$- \frac{1}{128(\ln\rho_2)^3}\ln\rho\,[8\rho_2^4 - 5\rho_2^4\ln\rho_2 - 16\,\rho_2^2\ln\rho_2 - 16\rho_2^2 + 8(\ln\rho_2)^3 + 20(\ln\rho_2)^2 +$$

$$+21\ln\rho_2 + 8]; \ldots; \ldots; \ldots \; (79)$$

$$P_{1,2} = \frac{1}{4\ln\rho_2}[\ln\rho + \rho^2\ln\rho - \rho^2 + 1] - \frac{1}{4(\ln\rho_2)^2}\ln\rho\,[\ln\rho_2 + \rho_2^2\ln\rho_2 - \rho^2 + 1]; \; (80)$$

$$P_{2,2} = \frac{1}{128(\ln\rho_2)^3}\ln\rho\,[6\rho_2^4(\ln\rho_2)^2 - 13\rho_2^4\ln\rho_2 + 8\rho_2^4 + 8\rho_2^2(\ln\rho_2)^2 - 16\rho_2^2 +$$

$$+6(\ln\rho_2)^2 + 13\ln\rho_2 + 8] -$$

$$- \frac{1}{128(\ln\rho_2)^2}[5\ln\rho_2 + 8\ln\rho + 8\rho_2^2\ln\rho_2 - 8\rho_2^2\ln\rho - 8\rho^2\ln\rho_2 + 3\rho^4\ln\rho_2 +$$

$$+8\rho^2\ln\rho - 8\rho_2^2 - 8\rho^2 + 8\rho_2^2\,\rho^2 + 6\ln\rho_2\ln\rho - 8\rho_2^2\rho^2\ln\rho_2 - 8\rho_2^2\rho^2\ln\rho + \; +8\rho_2^2\ln\rho_2\ln\rho -$$

$$2\rho^4\ln\rho_2\ln\rho + 8\rho_2^2\,\rho^2\ln\rho_2\ln\rho + 8]; \ldots; \ldots; \ldots \; (81)$$

The quasi-polynomials for solving the inverse unsteady heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow cylinder in a recurrent form are obtained as follows, based on the solutions obtained for the inverse unsteady heat conduc-tion problem when setting the boundary condition on the inner surface of the hollow cylinder, i.e. formulas (24) and (25). Here it is convenient to introduce local notation, which is valid only for this section, in order to avoid discrepancies in the solution of the problem in the future:

$$F_{n,1} \stackrel{\text{def}}{=} P_{n,1}\Big|^{(24)}; \; (82)$$

$$F_{n,2} \stackrel{\text{def}}{=} P_{n,2}\Big|^{(25)}. \; (83)$$

In other words, the functions Fn, 1 and Fn, 2 in the framework of this section denote the quasi-polynomials Pn, 1 and Pn, 2 for the inverse nonstationary heat conduction problem when the boundary condition on the inner surface of the hollow cylinder is specified from formulas (24) and (25), respectively. First we solve the problem for Pn, 2, since it is simpler than for Pn, 1; solving the first problem will be the basis for solving the second problem. It's obvious that:

$$P_{0,2} = \frac{F_{0,2}}{\ln\rho_2}. \; (84)$$

We rewrite the anti-Laplacians P1,2 in the following form:

$$P_{1,2} = \frac{1}{\ln\rho_2}\int_1^{\rho} \frac{1}{\rho}\int_1^{\rho} \rho\ln\rho\, d\rho d\rho - \frac{\ln\rho}{(\ln\rho_2)^2}\int_1^{\rho_2} \frac{1}{\rho}\int_1^{\rho} \rho\ln\rho\, d\rho d\rho =$$

$$= \frac{1}{\ln\rho_2}F_{1,2} - \frac{1}{(\ln\rho_2)^2}F_{0,2}F_{1,2}\Big|_{\rho=\rho_2} \; (85)$$

In order to obtain a recurrent solution to this problem, we rewrite the last expression as follows:

$$P_{1,2}=\frac{1}{\ln\rho_2}F_{1,2}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{1,2}\big|_{\rho=\rho_2}, \quad (86)$$

Where. Subsequent anti-Laplacians for quasi polynomials Pn, 2 will be as follows: $\Phi_{1,2}=\frac{1}{\ln\rho_2}F_{1,2}$

$$P_{2,2}=\frac{1}{\ln\rho_2}F_{2,2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{1,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{2,2}\big|_{\rho=\rho_2}, \quad (87)$$

Where; $\Phi_{2,2}=\frac{1}{\ln\rho_2}F_{2,2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{1,2}\big|_{\rho=\rho_2}$

$$P_{3,2}=\frac{1}{\ln\rho_2}F_{3,2}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{1,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{2,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{3,2}\big|_{\rho=\rho_2}, \quad (88)$$

where; $\Phi_{3,2}=\frac{1}{\ln\rho_2}F_{3,2}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{1,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{2,2}\big|_{\rho=\rho_2}$

$$P_{4,2}=\frac{1}{\ln\rho_2}F_{4,2}-\frac{1}{\ln\rho_2}F_{3,2}\Phi_{1,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{2,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{3,2}\big|_{\rho=\rho_2}-$$
$$-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{4,2}\big|_{\rho=\rho_2}, \ldots; \ldots; \quad (89)$$

Where. Consequently, n-th degree anti-Laplacians for quasi-polynomials Pn, 2 can be written in the following form:

$$\Phi_{4,2}=\frac{1}{\ln\rho_2}F_{4,2}-\frac{1}{\ln\rho_2}F_{3,2}\Phi_{1,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{2,2}\big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{3,2}\big|_{\rho=\rho_2}$$

$$P_{n,2}=\frac{1}{\ln\rho_2}F_{n,2}-\sum_{i=0}^{n-1}\frac{1}{\ln\rho_2}F_{n-1-i,2}\Phi_{i+1,2}\big|_{\rho=\rho_2}. \quad (90)$$

As can be seen from the formula (90), for its solution it is used as a "direct" recurrence, i.e. the use of the previous members of the series for the current member of a series, and "partial" recurrence, i.e. use in the output for the current member of a part of the same member of the series. Now the func-tion Φi, 2 should be determined. For this, a form should be formalized for them. Let us rewrite equation (86) for Φ1,2 in the form typical for larger values of the parameter i, namely:

$$\Phi_{1,2}=\frac{1}{\ln\rho_2}F_{1,2}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{0,2}\big|_{\rho=\rho_2}\ldots \quad (91)$$

For the expression Φ1,2 from formula (91) to be identically equal to its definition from (86), it is necessary (since F0,2= ln ρ) so that:

... (92) $\Phi_{0,2}\big|_{\rho=\rho_2}=0$

After the last formalization, we can write a closed expression for Φi, 2:

$$\Phi_{i,2}=\frac{1}{\ln\rho_2}F_{i,2}-\sum_{k=0}^{i-1}\frac{1}{\ln\rho_2}F_{i-k,2}\Phi_{k,2}\big|_{\rho=\rho_2}. \quad (93)$$

Thus, expressions (90), (93), (92) give an exact solution to the quasi-polynomial problem for solving the inverse non-stationary heat conduction problem when setting the temperature boundary condi-tions on both boundary surfaces for a hollow cylinder in a recur-rent form. As can be seen, the solu-tion for the functions Φi, 2 formally contains terms with. Obviously, all these terms are absent, for example, in (84) - (90). This is quite natural, since in the accepted representation (93) for Φi, 2 these terms are fictitious and equal (since) to zero: In the solution for Pn, 2 (90) there are no terms with Φ0.2, but there are terms with F0.2. The solution for Φi, 2 (93) does not contain terms with F0,2, but formally there are terms with identically equal to zero.

$$F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2}.\ \Phi_{0,2}\big|_{\rho=\rho_2}=0F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2}\equiv 0.\ F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2}$$

Now you should get a solution for quasi-polynomials Pn, 1, using the above solution method and based on the already available solutions for Pn, 2, Fn, 1, Fn, 2. It's obvious that:

$$P_{0,1}=1-\frac{\ln\rho}{\ln\rho_2}=F_{0,1}-P_{0,2}. \quad (94)$$

Addition (75) and (76) for the first terms gives the following expression:

$$P_{1,1}+P_{1,2}=\int_1^{\rho}\frac{1}{\rho}\int_1^{\rho}\rho\left(1-\frac{\ln\rho}{\ln\rho_2}\right)d\rho d\rho-\frac{\ln\rho}{\ln\rho_2}\int_1^{\rho_2}\frac{1}{\rho}\int_1^{\rho}\rho\left(1-\frac{\ln\rho}{\ln\rho_2}\right)d\rho d\rho+$$

$$+\int_1^{\rho}\frac{1}{\rho}\int_1^{\rho}\rho\frac{\ln\rho}{\ln\rho_2}d\rho d\rho-\frac{\ln\rho}{\ln\rho_2}\int_1^{\rho_2}\frac{1}{\rho}\int_1^{\rho}\rho\frac{\ln\rho}{\ln\rho_2}d\rho d\rho=$$

$$=\int_1^{\rho}\frac{1}{\rho}\int_1^{\rho}\rho\,d\rho d\rho-\frac{\ln\rho}{\ln\rho_2}\int_1^{\rho_2}\frac{1}{\rho}\int_1^{\rho}\rho\,d\rho d\rho=F_{1,1}-\frac{1}{\ln\rho_2}F_{0,2}F_{1,1}\Big|_{\rho=\rho_2}\dots(95)$$

In what follows, for Pn, 1, we will proceed in the same way as in the solution for quasi-polynomials Pn, 2, namely:

$$P_{1,1}=F_{1,1}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{1,1}\Big|_{\rho=\rho_2}-P_{1,2}\ ,(96)$$

Where. Subsequent anti-Laplacians for quasi-polynomials Pn, 1 will be as follows:$\Phi_{1,1}=F_{1,1}$

$$P_{2,1}=F_{2,1}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{1,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{2,1}\Big|_{\rho=\rho_2}-P_{2,2}\ ,(97)$$

$$\text{where;}\Phi_{2,1}=\frac{1}{\ln\rho_2}F_{2,1}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{1,1}\Big|_{\rho=\rho_2}$$

$$P_{3,1}=F_{3,1}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{1,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{2,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{3,1}\Big|_{\rho=\rho_2}-P_{3,2},(98)$$

$$\text{Where;}\Phi_{3,1}=F_{3,1}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{1,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{2,1}\Big|_{\rho=\rho_2}$$

$$P_{4,1}=F_{4,1}-\frac{1}{\ln\rho_2}F_{3,2}\Phi_{1,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{2,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{3,1}\Big|_{\rho=\rho_2}-$$

$$-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{4,1}\Big|_{\rho=\rho_2}-P_{4,2}\ (99)$$

$$\text{Where.}\Phi_{4,1}=F_{4,1}-\frac{1}{\ln\rho_2}F_{3,2}\Phi_{1,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{2,2}\Phi_{2,1}\Big|_{\rho=\rho_2}-\frac{1}{\ln\rho_2}F_{1,2}\Phi_{3,1}\Big|_{\rho=\rho_2}$$

Hence, n-th degree anti-Laplacians for quasi-polynomials Pn, 1 can be written in the following form:

$$P_{n,1}=F_{n,1}-\sum_{i=0}^{n-1}\frac{1}{\ln\rho_2}F_{n-1-i,2}\Phi_{i+1,1}\Big|_{\rho=\rho_2}-P_{n,2}.\ (100)$$

Now the function Φi, 1 should be determined. For this, a form should be formalized for them. Let us rewrite Eq. (96) for Φ1,1 in the form typical for larger values of the parameter i, namely:

$$\Phi_{1,1}=\frac{1}{\ln\rho_2}F_{1,1}-\frac{1}{\ln\rho_2}F_{0,2}\Phi_{0,1}\Big|_{\rho=\rho_2}\dots(101)$$

For the expression Φ1,1 from formula (101) to be identically equal to its definition from (96), it is necessary (since F0,2= ln ρ) so that:

$$\dots(102)\Phi_{0,1}\Big|_{\rho=\rho_2}=0$$

After the last formalization, we can write a closed expression for Φi, 1:

$$\Phi_{i,1}=F_{i,1}-\sum_{k=0}^{i-1}\frac{1}{\ln\rho_2}F_{i-k,2}\Phi_{k,1}\Big|_{\rho=\rho_2}.\ (103)$$

Thus, expressions (100), (103), (102) give an exact solution to the quasi-polynomial problem for solving the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow cylinder in a recurrent form. As can be seen, the solution for the functions Φi, 1 formally contains terms with. Obviously, all these terms are absent, for example, in (94) - (103). This is quite natural, since in the accepted representation (103) for Φi, 1 these terms are fictitious and equal (since) to zero: In the solution for Pn, 1 (100), there are no terms with Φ0.1, but there are terms with F0.2. The solution for Φi, 1 (103) does not contain terms with F0,2, but formally there are terms with identically equal to zero. In principle, the problem of the ex-act solution of quasi-polynomials Pn, 2 - (90), (93), (92) - and Pn, 1 - (100), (103), (102) - for the in-verse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow cylinder in a recurrent form, it can be completed. However, it is possible to write these solutions in a combined form, for which it is necessary to rewrite in the ap-propriate form the series for Φi, 1 and Φi, 2 from formulas (103) and (93), respective-ly:

$$F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2}\cdot\Phi_{0,1}\big|_{\rho=\rho_2}=0F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2}\equiv 0.\,F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2}$$

$$\Phi_{i+1,1}=F_{i+1,1}-\sum_{k=0}^{i}\frac{1}{\ln\rho_2}F_{i+1-k,2}\Phi_{k,1}\big|_{\rho=\rho_2}.\ (104)$$

$$\Phi_{i+1,2}=\frac{1}{\ln\rho_2}F_{i+1,2}-\sum_{k=0}^{i}\frac{1}{\ln\rho_2}F_{i+1-k,2}\Phi_{k,2}\big|_{\rho=\rho_2}.\ (105)$$

In the combined form, the exact solutions of this problem (for Pn, 1 - (100) and for Pn, 2 - (90)) will look like this:

$$P_{n,1}=F_{n,1}-\sum_{i=0}^{n-1}\frac{1}{\ln\rho_2}F_{n-1-i,2}\left[F_{i+1,1}-\sum_{k=0}^{i}\frac{1}{\ln\rho_2}F_{i+1-k,2}\Phi_{k,1}\big|_{\rho=\rho_2}\right]\Bigg|_{\rho=\rho_2}-P_{n,2}.\ (106)$$

$$P_{n,2}=\frac{1}{\ln\rho_2}F_{n,2}-\sum_{i=0}^{n-1}\frac{1}{\ln\rho_2}F_{n-1-i,2}\left[\frac{1}{\ln\rho_2}F_{i+1,2}-\sum_{k=0}^{i}\frac{1}{\ln\rho_2}F_{i+1-k,2}\Phi_{k,2}\big|_{\rho=\rho_2}\right]\Bigg|_{\rho=\rho_2}.\ (107)$$

Hollow Ball (Solution without using Bernoulli Numbers Bn)

Quasi-polynomials for solving the inverse non-stationary heat conduction problem when specifying the temperature boundary conditions on both boundary surfaces for a hollow ball in recurrent form are obtained from the solutions obtained for the inverse non-stationary heat conduction problem when specifying the boundary condition on the inner surface of the hollow ball, i.e. formulas (36) and (37). Solutions are obtained in the same way as for a flat plate (see 3.1.1) or for a hollow cylin-der (see 3.1.2). Quasi-polynomials Pn, 1 and Pn, 2 for a hollow ball will be as follows:

$$P_{n+1,1}=\int_1^{\rho}\frac{1}{\rho^2}\int_1^{\rho}\rho^2P_{n,1}\,d\rho d\rho-\frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^{\rho}\rho^2P_{n,1}\,d\rho d\rho;\ (108)$$

$$P_{n+1,2}=\int_1^{\rho}\frac{1}{\rho^2}\int_1^{\rho}\rho^2P_{n,2}\,d\rho d\rho-\frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^{\rho}\rho^2P_{n,2}\,d\rho d\rho;\ (109)$$

$$P_{0,1}=\frac{1}{(\rho_2-1)}\left(\frac{\rho_2}{\rho}-1\right);\ P_{0,2}=\frac{\rho_2}{(\rho_2-1)}\left(1-\frac{1}{\rho}\right).\ (110)$$

For the first quasi-polynomials P1,1 and P2,1, P1,2 and P2,2, etc. for a hollow ball, you can write:

$$P_{1,1}=\frac{4}{3}\frac{1}{\rho}(\rho_2-1)^2\left(-\frac{1}{4}\frac{(\rho-1)}{(\rho_2-1)}+\frac{3}{8}\frac{(\rho-1)^2}{(\rho_2-1)^2}-\frac{1}{8}\frac{(\rho-1)^3}{(\rho_2-1)^3}\right);\ (111)$$

$$P_{2,1}=\frac{4}{15}\frac{1}{\rho}(\rho_2-1)^4\left(\frac{1}{12}\frac{(\rho-1)}{(\rho_2-1)}-\frac{5}{24}\frac{(\rho-1)^3}{(\rho_2-1)^3}+\frac{5}{32}\frac{(\rho-1)^4}{(\rho_2-1)^4}-\frac{1}{32}\frac{(\rho-1)^5}{(\rho_2-1)^5}\right);\ (112)$$

$$P_{3,1}=\frac{8}{315}\frac{1}{\rho}(\rho_2-1)^6\left(-\frac{1}{12}\frac{(\rho-1)}{(\rho_2-1)}+\frac{7}{48}\frac{(\rho-1)^3}{(\rho_2-1)^3}-\frac{7}{64}\frac{(\rho-1)^5}{(\rho_2-1)^5}+\frac{7}{128}\frac{(\rho-1)^6}{(\rho_2-1)^6}-\frac{1}{128}\frac{(\rho-1)^7}{(\rho_2-1)^7}\right);\ (113)$$

$$P_{4,1}=\frac{4}{2835}\frac{1}{\rho}(\rho_2-1)^8\left(\frac{3}{20}\frac{(\rho-1)}{(\rho_2-1)}-\frac{1}{4}\frac{(\rho-1)^3}{(\rho_2-1)^3}+\frac{21}{160}\frac{(\rho-1)^5}{(\rho_2-1)^5}-\frac{3}{64}\frac{(\rho-1)^7}{(\rho_2-1)^7}+\frac{9}{512}\frac{(\rho-1)^8}{(\rho_2-1)^8}-\right.$$
$$\left.-\frac{1}{512}\frac{(\rho-1)^9}{(\rho_2-1)^9}\right);\ \dots;\ \dots;\ (114)$$

$$P_{1,2}=\frac{4}{3}\frac{\rho_2}{\rho}(\rho_2-1)^2\left(-\frac{1}{4}\frac{(\rho_2-\rho)}{(\rho_2-1)}+\frac{3}{8}\frac{(\rho_2-\rho)^2}{(\rho_2-1)^2}-\frac{1}{8}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}\right);\ (115)$$

$$P_{2,2}=\frac{4}{15}\frac{\rho_2}{\rho}(\rho_2-1)^4\left(\frac{1}{12}\frac{(\rho_2-\rho)}{(\rho_2-1)}-\frac{5}{24}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}+\frac{5}{32}\frac{(\rho_2-\rho)^4}{(\rho_2-1)^4}-\frac{1}{32}\frac{(\rho_2-\rho)^5}{(\rho_2-1)^5}\right);\ (116)$$

$$P_{3,2}=\frac{8}{315}\frac{\rho_2}{\rho}(\rho_2-1)^6\left(-\frac{1}{12}\frac{(\rho_2-\rho)}{(\rho_2-1)}+\frac{7}{48}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}-\frac{7}{64}\frac{(\rho_2-\rho)^5}{(\rho_2-1)^5}+\frac{7}{128}\frac{(\rho_2-\rho)^6}{(\rho_2-1)^6}-\right.$$
$$\left.\frac{1}{128}\frac{(\rho_2-\rho)^7}{(\rho_2-1)^7}\right);(117)$$

$$P_{4,2}=\frac{4}{2835}\frac{\rho_2}{\rho}(\rho_2-1)^8\left(\frac{3}{20}\frac{(\rho_2-\rho)}{(\rho_2-1)}-\frac{1}{4}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}+\frac{21}{160}\frac{(\rho_2-\rho)^5}{(\rho_2-1)^5}-\frac{3}{64}\frac{(\rho_2-\rho)^7}{(\rho_2-1)^7}+\right.$$
$$\left.+\frac{9}{512}\frac{(\rho_2-\rho)^8}{(\rho_2-1)^8}-\frac{1}{512}\frac{(\rho_2-\rho)^9}{(\rho_2-1)^9}\right);\ \dots;\ \dots;(118)$$

Here it is convenient to introduce local notation, which is valid only for this section, in order to avoid discrepancies in the solution of the problem in the future:

$F_{n,1} \stackrel{\text{def}}{=} P_{n,1}\Big|^{(36)}$; (119)

$F_{n,2} \stackrel{\text{def}}{=} P_{n,2}\Big|^{(37)}$. (120)

In other words, the functions Fn, 1 and Fn, 2 within the framework of this section denote the quasi-polynomials Pn, 1 and Pn, 2 for the inverse nonstationary heat conduction problem when the boundary condition on the inner surface of the hollow ball is specified from formulas (36) and (37), respectively. First we solve the problem for Pn, 2, since it is simpler than for Pn, 1; the solution of the first problem will be taken as a basis for solving the second problem. We rewrite the anti-Laplacians P1,2 in the following form:

$$P_{1,2} = \int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2 P_{0,2}\, d\rho d\rho - \frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho \rho^2 P_{0,2}\, d\rho d\rho =$$

$$= \int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2 \frac{\rho_2}{(\rho_2-1)}\left(1-\frac{1}{\rho}\right) d\rho d\rho - \frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho \rho^2 \frac{\rho_2}{(\rho_2-1)}\left(1-\frac{1}{\rho}\right) d\rho d\rho ==$$

$$\frac{\rho_2}{(\rho_2-1)}\int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2\left(1-\frac{1}{\rho}\right) d\rho d\rho - \left(1-\frac{1}{\rho}\right)\left(\frac{\rho_2}{(\rho_2-1)}\right)^2\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho \rho^2\left(1-\frac{1}{\rho}\right) d\rho d\rho =$$

$$= \frac{\rho_2}{(\rho_2-1)}F_{1,2} - F_{0,2}\left(\frac{\rho_2}{(\rho_2-1)}\right)^2 F_{1,2}\Big|_{\rho=\rho_2}. \quad (121)$$

In order to obtain a recurrent solution to this problem, we rewrite the last expression as follows:

$$P_{1,2} = \frac{\rho_2}{(\rho_2-1)}F_{1,2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{1,2}\Big|_{\rho=\rho_2}, \quad (122)$$

Where. Subsequent anti-Laplacians for quasi polynomials Pn, 2 will be as follows: $\Phi_{1,2} = \frac{\rho_2}{(\rho_2-1)}F_{1,2}$

$$P_{2,2} = \frac{\rho_2}{(\rho_2-1)}F_{2,2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{1,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{2,2}\Big|_{\rho=\rho_2}, \quad (123)$$

where; $\Phi_{2,2} = \frac{\rho_2}{(\rho_2-1)}F_{2,2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{1,2}\Big|_{\rho=\rho_2}$

$$P_{3,2} = \frac{\rho_2}{(\rho_2-1)}F_{3,2} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{1,2}\Big|_{\rho=\rho_2} -$$

$$- \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{2,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{3,2}\Big|_{\rho=\rho_2}, \quad (124)$$

where; $\Phi_{3,2} = \frac{\rho_2}{(\rho_2-1)}F_{3,2} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{1,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{2,2}\Big|_{\rho=\rho_2}$

$$P_{4,2} = \frac{\rho_2}{(\rho_2-1)}F_{4,2} - \frac{\rho_2}{(\rho_2-1)}F_{3,2}\Phi_{1,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{2,2}\Big|_{\rho=\rho_2} - \text{, ...; ...; (125)}$$

$$- \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{3,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{4,2}\Big|_{\rho=\rho_2}$$

Where; $\Phi_{4,2} = \frac{\rho_2}{(\rho_2-1)}F_{4,2} - \frac{\rho_2}{(\rho_2-1)}F_{3,2}\Phi_{1,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{2,2}\Big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{3,2}\Big|_{\rho=\rho_2}\cdots$

Hence, n-th degree anti-Laplacians for quasi-polynomials Pn, 2 can be written in the following form:

$$P_{n,2} = \frac{\rho_2}{(\rho_2-1)}F_{n,2} - \sum_{i=0}^{n-1}\frac{\rho_2}{(\rho_2-1)}F_{n-1-i,2}\Phi_{i+1,2}\Big|_{\rho=\rho_2}. \quad (126)$$

As can be seen from the formula (126), for its solution it is used as a "direct" recurrence, i.e. the use of the previous members of the series for the current member of a series, and "partial" recurrence, i.e. use in the output for the current member of a part of the same member of the series. Now the func-tion Φi, 2 should be determined. For this, a form should be formalized for them. Let us rewrite Eq. (122) for Φ1,2 in the form typical for larger values of the parameter i, namely:

$$\Phi_{1,2} = \frac{\rho_2}{(\rho_2-1)}F_{1,2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{0,2}\Big|_{\rho=\rho_2}\ldots \quad (127)$$

For the expression Φ1,2 from formula (127) to be identically equal to its definition from (122), it is necessary (since F0,2= 1 - 1 / ρ) so that:

$$\Phi_{0,2}\big|_{\rho=\rho_2} = 0... (128)$$

After the last formalization, we can write a closed expression for Φi, 2:

$$\Phi_{i,2} = \frac{\rho_2}{(\rho_2-1)}F_{i,2} - \sum_{k=0}^{i-1}\frac{\rho_2}{(\rho_2-1)}F_{i-k,2}\Phi_{k,2}\big|_{\rho=\rho_2}. (129)$$

Thus, expressions (126), (129), (128) give an exact solution to the quasi-polynomial problem for solving the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow ball in a recurrent form. As can be seen, in the solution for the functions Φi, 2formally, terms with are present. Obviously, all these terms are absent, for example, in (121) - (126). This is quite natural, since in the accepted representation (93) for Φi, 2 these terms are fictitious and equal (since) to zero: In the solution for Pn, 2 (126), there are no terms with Φ0.2, but there are terms with F0.2. The solution for Φi, 2 (129) lacks terms with F0,2, but formally there are terms with identically equal to zero. Now it is necessary to obtain a solution for the quasi-polynomials Pn, 1, using the above solution method and based on the already available solutions for Pn, 2, Fn, 1, Fn, 2. It's obvious that: $F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2}.\ \Phi_{0,2}\big|_{\rho=\rho_2} = 0 F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2} \equiv 0.\ F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2}$

$$P_{0,1} + P_{0,2} = \frac{1}{\rho_2-1}\frac{\rho_2-\rho}{\rho} + \frac{\rho_2}{\rho_2-1}\frac{\rho-1}{\rho} = 1. (130)$$

Addition (108) and (109) for the first terms gives the following expression:

$$P_{1,1} + P_{1,2} = \int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2 \frac{1}{\rho_2-1}\frac{\rho_2-\rho}{\rho} d\rho d\rho - \frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho \rho^2\frac{1}{\rho_2-1}\frac{\rho_2-\rho}{\rho}d\rho d\rho +$$

$$+\int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2 \frac{\rho_2}{\rho_2-1}\frac{\rho-1}{\rho} d\rho d\rho - \frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho \rho^2\frac{\rho_2}{\rho_2-1}\frac{\rho-1}{\rho}d\rho d\rho =$$

$$... (131)= \int_1^\rho \frac{1}{\rho^2}\int_1^\rho \rho^2\, d\rho d\rho - \frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho \rho^2\, d\rho d\rho = F_{1,1} - \frac{\rho_2}{\rho_2-1}F_{0,2}F_{1,1}\big|_{\rho=\rho_2}$$

In order to obtain an expression for Pn, 1 for a recurrent solution of this problem, we rewrite the last expression in the same way as when solving for quasi-polynomials Pn, 2, namely:

$$P_{1,1} = F_{1,1} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{1,1}\big|_{\rho=\rho_2} - P_{1,2}, (132)$$

Where. Subsequent anti-Laplacians for quasi-polynomials Pn, 1 will be as follows: $\Phi_{1,1} = F_{1,1}$

$$P_{2,1} = F_{2,1} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{1,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{2,1}\big|_{\rho=\rho_2} - P_{2,2}\ , (133)$$

Where; $\Phi_{2,1} = F_{2,1} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{1,1}\big|_{\rho=\rho_2}$

$$P_{3,1} = F_{3,1} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{1,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{2,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{3,1}\big|_{\rho=\rho_2} - P_{3,2}. (134)$$

Where; $\Phi_{3,1} = F_{3,1} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{1,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{2,1}\big|_{\rho=\rho_2}$

$$P_{4,1} = F_{4,1} - \frac{\rho_2}{(\rho_2-1)}F_{3,2}\Phi_{1,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{2,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{3,1}\big|_{\rho=\rho_2} -,...,..., (135)$$

$$-\frac{\rho_2}{(\rho_2-1)}F_{0,2}\Phi_{4,1}\big|_{\rho=\rho_2} - P_{4,2}$$

Where. $\Phi_{4,1} = F_{4,1} - \frac{\rho_2}{(\rho_2-1)}F_{3,2}\Phi_{1,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{2,2}\Phi_{2,1}\big|_{\rho=\rho_2} - \frac{\rho_2}{(\rho_2-1)}F_{1,2}\Phi_{3,1}\big|_{\rho=\rho_2}$

Hence, n-th degree anti-Laplacians for quasi-polynomials Pn, 1 can be written in the following form:

$$P_{n,1} = F_{n,1} - \sum_{i=0}^{n-1}\frac{\rho_2}{(\rho_2-1)}F_{n-1-i,2}\Phi_{i+1,1}\big|_{\rho=\rho_2} - P_{n,2}. (136)$$

Now the function Φi, 1 should be determined. For this, a form should be formalized for them. Let us rewrite Eq. (132) for Φ1,1 in a form typical for larger values of the parameter i, namely:

$$\Phi_{1,1} = F_{1,1} - \frac{\rho_2}{(\rho_2-1)} F_{0,2}\Phi_{0,1}\big|_{\rho=\rho_2} \ldots (137)$$

For the expression Φ1,1 from formula (137) to be identically equal to its definition from (132), it is necessary (since F0,2= 1 - 1 / ρ) so that:

$$\Phi_{0,1}\big|_{\rho=\rho_2} = 0 \ldots (138)$$

After the last formalization, we can write a closed expression for Φi, 1:

$$\Phi_{i,1} = F_{i,1} - \sum_{k=0}^{i-1} \frac{\rho_2}{(\rho_2-1)} F_{i-k,2}\Phi_{k,1}\big|_{\rho=\rho_2}. (139)$$

Thus, expressions (136), (139), (138) give an exact solution to the quasi-polynomial problem for solving the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow ball in a recurrent form. As can be seen, in the solution for the functions Φi,1formally, terms with are present. Obviously, all these terms are absent, for example, in (131) - (139). This is quite natural, since in the accepted representation (139) for Φi, 1 these terms are fictitious and equal (since) to zero: In the solution for Pn, 1 (136) there are no terms with Φ0.1, but there are terms with F0.2. The solution for Φi, 1 (139) lacks terms with F0,2, but formally there are terms with identically zero. In principle, the problem of the exact solution of quasi-polynomials Pn, 2 - (126), (129), (128) - and Pn, 1 - (136), (139), (138) - for the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow ball in recurrent form can be completed. However, it is possible to write these solutions in a combined form, for which it is necessary to rewrite the series for Φi, 1 and

$$\Phi\mathrm{i}, F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2}. \Phi_{0,1}\big|_{\rho=\rho_2} = 0 F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2} \equiv 0. F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2}$$

$$\Phi_{i+1,1} = F_{i+1,1} - \sum_{k=0}^{i} \frac{\rho_2}{(\rho_2-1)} F_{i+1-k,2}\Phi_{k,1}\big|_{\rho=\rho_2}. (140)$$

$$\Phi_{i+1,2} = \frac{\rho_2}{(\rho_2-1)} F_{i+1,2} - \sum_{k=0}^{i} \frac{\rho_2}{(\rho_2-1)} F_{i+1-k,2}\Phi_{k,2}\big|_{\rho=\rho_2}. (141)$$

In the combined form, the exact solutions of this problem (for Pn, 1 - (136) and for Pn, 2 - (126)) will look like this:

$$P_{n,1} = F_{n,1} - $$

$$-\sum_{i=0}^{n-1} \frac{\rho_2}{(\rho_2-1)} F_{n-1-i,2}\left[F_{i+1,1} - \sum_{k=0}^{i} \frac{\rho_2}{(\rho_2-1)} F_{i+1-k,2}\Phi_{k,1}\big|_{\rho=\rho_2}\right]\Bigg|_{\rho=\rho_2} - P_{n,2}. (142)$$

$$P_{n,2} = \frac{\rho_2}{(\rho_2-1)} F_{n,2} - $$

$$-\sum_{i=0}^{n-1} \frac{\rho_2}{(\rho_2-1)} F_{n-1-i,2}\left[\frac{\rho_2}{(\rho_2-1)} F_{i+1,2} - \sum_{k=0}^{i} \frac{\rho_2}{(\rho_2-1)} F_{i+1-k,2}\Phi_{k,2}\big|_{\rho=\rho_2}\right]\Bigg|_{\rho=\rho_2}. (143)$$

Solutions in a Recurrent Form for a Nonstationary Linear Inverse Heat Conduction Problem for Flat Plates and Hollow Balls (Bodies of One-Dimensional Geometry) with Boundary Temperature Conditions on Both Surfaces using Bernoulli Numbers

IN In Sections 3.1.1 and 3.1.3, solutions were obtained in recurrent form for the nonstationary linear inverse problem of heat conduction for flat plates and hollow balls, i.e. bodies of one-dimensional geometry, with boundary temperature conditions on both surfaces using the method of mathematical induction without using Bernoulli numbers Bn. This section poses the problem of obtaining recurrent solutions using the Bernoulli numbers Bn.

Flat Plate (Solution using Bernoulli Numbers Bn)

In 3.1.1, the quasi-polynomials Pn, 1 and Pn, 2 were solved for a flat plate: (40) - (50). Therefore, using the method of mathematical induction, one can write quasi-polynomials for solving the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a flat plate in recurrent form when using Bernoulli numbers Bn:

$$P_{n,1} = P_{n-1,1} - \frac{1}{(2n+1)!}\rho^{2n+1} + \frac{1}{(2n)!}\rho^{2n} +$$

$$+\sum_{k=0}^{2n-1} \frac{2^{2n+1-k}}{k!\,(2n-1-k)!}\left(\frac{B_{2n-1-k}}{4} - \frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right)\rho^k, \ (144)$$

$$P_{n,2} = P_{n-1,2} - \sum_{k=0}^{2n+1} \frac{(-1)^{4n+2-k}}{k!\,(2n+1-k)!}\rho^k + \sum_{k=0}^{2n} \frac{(-1)^{4n-k}}{k!\,(2n-k)!}\rho^k + \ (145)$$

$$+\sum_{k=0}^{2n-1}\sum_{l=0}^{k} \frac{(-1)^{4k-l}}{l!\,(k-l)!}\frac{2^{2n+1-k}}{(2n-1-k)!}\left(\frac{B_{2n-1-k}}{4} - \frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right)\rho^l,$$

Where Bn are Bernoulli numbers: (is the binomial coefficient; the number of combinations from N to K) [25]. For example, the first few Bernoulli numbers are equal: B0 = 1; B1 = –1 / 2; B2 = 1/6; B3 = 0; B4 = –1 / 30; B5 = 0; B6 = 1/42; B7 = 0; B8 = –1 / 30; B9 = 0; B10 = 5/66; B11 = 0; B12 = -691 / 2730; B13 = 0; B14 = 7/6; B15 = 0; B16 = -3617 / 510; B17 = 0; B18 = 43867/798; B19 = 0; B20 = –174611 / 330 ... $B_n = \frac{-1}{n+1}\sum_{k=1}^{n} C_{k+1}^{n+1} B_{n-1}, n \in \mathbb{N} C_N^K = \frac{N!}{K!(N-K)!}$

For the last quasi-polynomial Pn, 2, one can rearrange and write it in the following form:

$$P_{n,2} = P_{n-1,2} - \frac{1}{(2n+1)!}(1-\rho)^{2n+1} + \frac{1}{(2n)!}(1-\rho)^{2n} + \ (146)$$

$$+\sum_{k=0}^{2n-1} \frac{2^{2n+1-k}}{k!\,(2n-1-k)!}\left(\frac{B_{2n-1-k}}{4} - \frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right)(1-\rho)^k.$$

Hollow Ball (Solution using Bernoulli Numbers Bn)

In 3.1.3, quasi-polynomials Pn, 1 and Pn, 2 were obtained for a hollow ball: (108) - (118). Therefore, using the method of mathematical induction, it is possible to write quasi-polynomials for solving the inverse non-stationary heat conduction problem when setting the temperature boundary conditions on both boundary surfaces for a hollow ball in recurrent form when using Bernoulli numbers Bn:

$$P_{n,1} = P_{n-1,1} - \frac{1}{(2n+1)!}\frac{1}{(\rho_2-1)}\frac{1}{\rho}(\rho-1)^{2n+1} + \frac{1}{(2n)!}\frac{1}{\rho}(\rho-1)^{2n} +$$

$$+\frac{1}{\rho}\sum_{k=0}^{2n-1} \frac{2^{2n+1-k}}{k!\,(2n-1-k)!}(\rho_2-1)^{2n-2-k}(\rho-1)^k \times \ (147)$$

$$\times\left(\frac{B_{2n-1-k}}{4} - (\rho_2-1)^2\frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right);$$

$$P_{n,2} = P_{n-1,2} - \frac{1}{(2n+1)!}\frac{1}{(\rho_2-1)}\frac{\rho_2}{\rho}(\rho_2-\rho)^{2n+1} + \frac{1}{(2n)!}\frac{\rho_2}{\rho}(\rho_2-\rho)^{2n} +$$

$$+\frac{\rho_2}{\rho}\sum_{k=0}^{2n-1} \frac{2^{2n+1-k}}{k!\,(2n-1-k)!}(\rho_2-1)^{2n-2-k}(\rho_2-\rho)^k \times \ (148)$$

$$\times\left(\frac{B_{2n-1-k}}{4} - (\rho_2-1)^2\frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right).$$

For the given unsteady temperature boundary conditions on both surfaces Θn, 1 and Θn, 2, the re-currence relations are:

$$\Theta_{n,i} = \frac{r_1^2}{a}\frac{\partial\Theta_{n-1,i}}{\partial\tau}, \forall\, i = 1,2\ldots \ (149)$$

Conclusions

1. The relevance of the problem of solving the inverse linear non-stationary problem of heat conduction of a one-dimensional geometric shape, obtained in this work in a closed recurrent form, lies in the fact that it is possible with a sufficient degree of accuracy to reconstruct the boundary conditions from the measurements of the heat flow sensor.
2. In this paper, we obtain exact analytical solutions for a nonstationary linear inverse problem of heat conduction for bodies of one-dimensional geometry with boundary conditions on one surface, as well as on two surfaces for a flat body and hollow cylinders and spheres, obtained in recurrent form. The solutions under the boundary conditions on two surfaces for a plane body and for a hollow ball were obtained both with and without the use of Bernoulli numbers.
3. The recurrent form of writing the solution of the non-station-

ary linear inverse problem of heat conduction obtained in the work for bodies of one-dimensional geometry with boundary conditions on one surface, as well as on two surfaces for a flat body, a hollow cylinder and a hollow sphere, is a solution in a closed form from a single position, that not always explicitly possible.

4. From a practical point of view, the solutions obtained can be used to calculate nonstationary fields of temperatures and heat flux densities for various materials used in aviation and rocket and space technology, based on the measured nonstationary boundary conditions on one of the sides, as well as on two surfaces for flat body, hollow cylinder and hollow sphere.

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