ISSN: 2634 - 8853

Journal of Engineering and Applied Sciences Technology

SCIENTIFIC Research and Community

Short Communication

Open Access

'Clifford Algebra' Complex Numbers For 3-D Vector Representation

Gerald N Pellegrini

Independent Researcher, USA

ABSTRACT
The use of quaternions has historically been an approach to extend complex numbers to represent 3-D vectors. This quaternion representation is not unique. It is shown that an alternative approach to such an extension is to define complex numbers obeying a Clifford algebra different from those of quaternions. Such an algebra is not new but appears to be preferable and less likely to cause confusion.

***Corresponding author**
Gerald N Pellegrini, Independent Researcher, USA.

Received: December 26, 2024; **Accepted:** December 28, 2024; **Published:** June 04, 2025

Introduction
The traditional approach to extending complex numbers to represent 3-D vectors is to define 'imaginary' numbers i, j, k according to the well-known Hamiltonian 'quaternion' definition [1]:

$$i^2 = j^2 = k^2 = ijk = -1\,, \tag{1}$$

with 3-D 'vectors' being described as 'complex' numbers of the form[1]

$$\boldsymbol{v} = v_1 i + v_2 j + v_3 k\,.$$

However, as shown below there is an alternative definition of 'complex' numbers which may represent 3-D vectors according to a Clifford algebra where[2]

$$i^2 = j^2 = k^2 = 1 \tag{2}$$

$$ijk = I \qquad I^2 = -1 \tag{3}$$

Rotations according to 'Quaternion' Definition
According to the quaternion definition Eq. (1), the product of 2 vectors **u** and $\boldsymbol{v}$, where

$$\boldsymbol{u} = u_1 i + u_2 j + u_3 k$$

$$\boldsymbol{v} = v_1 i + v_2 j + v_3 k$$

is

$$\boldsymbol{uv} = -\boldsymbol{u}\cdot\boldsymbol{v} + (\boldsymbol{u}\times\boldsymbol{v})\,. \tag{4}$$

For any unit vector $\boldsymbol{n}$

$$\boldsymbol{nn} = -1\,, \tag{5}$$

Using the quaternion definition rotation of vector $\boldsymbol{R}$ about an axis $\boldsymbol{n}$ by angle θ is equal to

$$\boldsymbol{R}' = \boldsymbol{qRq}^{-1}$$

where

$$\boldsymbol{q} = cos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\boldsymbol{n}$$

$$\boldsymbol{q}^{-1} = cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\boldsymbol{n}$$

$$\boldsymbol{R}' = \left(cos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right)\boldsymbol{R}\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right).$$

Expanding

$$\boldsymbol{R}' = cos\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)\boldsymbol{R} + sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)(\boldsymbol{nR} - \boldsymbol{Rn}) - sin\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)\boldsymbol{nRn} \tag{6}$$

From quaternion vector product Eq. (4),

[1]Historically confusion regarding quaternion complex number representation of 3-D vectors was the main reason for vector analysis, as it is known today, to be the preferable approach.

[2] It is noted that this definition of i, J, k can be represented by the Pauli spin matrices respectively. However, it is not necessary that these complex number symbols be given any 'representation', since it is only their multiplicative properties that matter.

$$\boldsymbol{nR} = -\boldsymbol{n}\cdot\boldsymbol{R} + (\boldsymbol{n}\times\boldsymbol{R})$$

$$\boldsymbol{Rn} = -\boldsymbol{n}\cdot\boldsymbol{R} - (\boldsymbol{n}\times\boldsymbol{R})$$

$$\boldsymbol{nR} - \boldsymbol{Rn} = 2(\boldsymbol{n}\times\boldsymbol{R})\,, \qquad (7)$$

Also,

$$\boldsymbol{nR}_{\parallel} = \boldsymbol{R}_{\parallel}\boldsymbol{n}$$

$$\boldsymbol{nR}_{\perp} = -\boldsymbol{R}_{\perp}\boldsymbol{n}$$

where $\boldsymbol{R}_{\parallel}$ and $\boldsymbol{R}_{\perp}$ are the components of ***R*** parallel and perpendicular to ***n*** respectively.

Using these relation along with Eq. (5) we then have

$$\boldsymbol{nRn} = \boldsymbol{n}(\boldsymbol{R}_{\parallel} + \boldsymbol{R}_{\perp})\boldsymbol{n} = \boldsymbol{R}_{\parallel}\boldsymbol{nn} - \boldsymbol{R}_{\perp}\boldsymbol{nn} = \boldsymbol{R}_{\perp} - \boldsymbol{R}_{\parallel}$$

$$\boldsymbol{R} - 2\boldsymbol{R}_{\parallel}\,. \qquad (8)$$

Using Eqs. (7) and (8) in Eq. (6) gives

$$\boldsymbol{R}' = cos\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)\boldsymbol{R} - sin\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)(\boldsymbol{R} - 2\boldsymbol{R}_{\parallel}) + 2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)(\boldsymbol{n}\times\boldsymbol{R})\,,$$

Using half angle identities

$$\boldsymbol{R}' = \boldsymbol{R}cos(\theta) + 2\,sin\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)\boldsymbol{R}_{\parallel} + sin(\theta)(\boldsymbol{n}\times\boldsymbol{R})$$

$$\boldsymbol{R}' = \boldsymbol{R}cos(\theta) + \big(1 - cos(\theta)\big)\boldsymbol{R}_{\parallel} + sin(\theta)(\boldsymbol{n}\times\boldsymbol{R})\,.$$

This is the well-known Rodrigues formula for axis angle rotation.

Also, according to vector multiplication Eq. (4)

$$-\boldsymbol{uv} = \boldsymbol{u}\cdot\boldsymbol{v} - (\boldsymbol{u}\times\boldsymbol{v})$$

$$-\boldsymbol{vu} = \boldsymbol{u}\cdot\boldsymbol{v} + (\boldsymbol{u}\times\boldsymbol{v})\,.$$

If ***u*** and ***v*** are both unit vectors separated by angle $\frac{\theta}{2}$

$$\boldsymbol{q} = cos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\boldsymbol{n} = -\boldsymbol{vu} = \boldsymbol{u}\cdot\boldsymbol{v} + (\boldsymbol{u}\times\boldsymbol{v})$$

$$\boldsymbol{q}^{-1} = cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\boldsymbol{n} = -\boldsymbol{uv} = \boldsymbol{u}\cdot\boldsymbol{v} - (\boldsymbol{u}\times\boldsymbol{v})\,,$$

where the axis of rotation ***n*** is the unit vector in the ($\boldsymbol{u}\times\boldsymbol{v}$) direction.

Rotation of vector ***R*** is then the vector product

$$\boldsymbol{R}' = \boldsymbol{qRq}^{-1} = -\boldsymbol{vuR}(-\boldsymbol{uv}) = \boldsymbol{vuRuv}\,. \qquad (9)$$

Rotations according to 'Clifford Algebra' Definition

According to the Clifford algebra definition Eqs. (2) and (3), the product of 2 vectors ***u*** and ***v***

$$\boldsymbol{u} = u_1 i + u_2 j + u_3 k$$

$$\boldsymbol{v} = v_1 i + v_2 j + v_3 k$$

is

$$\boldsymbol{uv} = \boldsymbol{u}\cdot\boldsymbol{v} + I(\boldsymbol{u}\times\boldsymbol{v})\,. \qquad (10)$$

For any unit vector ***n***

$$\boldsymbol{nn} = 1\,. \qquad (11)$$

Using the Clifford algebra definition rotation of vector ***R*** about an axis ***n*** by angle θ is equal to

$$\boldsymbol{R}' = \left(cos\left(\frac{\theta}{2}\right) - Isin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right)\boldsymbol{R}\left(cos\left(\frac{\theta}{2}\right) + Isin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right)\,.$$

Expanding using $I^2 = -1$

$$\boldsymbol{R}' = cos\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)\boldsymbol{R} + Isin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)(\boldsymbol{Rn} - \boldsymbol{nR}) +$$

$$sin\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)\boldsymbol{nRn}\,. \qquad (12)$$

From Clifford algebra vector product Eq. (10)

$$\boldsymbol{nR} = \boldsymbol{n}\cdot\boldsymbol{R} + I(\boldsymbol{n}\times\boldsymbol{R})$$

$$\boldsymbol{Rn} = \boldsymbol{n}\cdot\boldsymbol{R} - I(\boldsymbol{n}\times\boldsymbol{R})$$

$$\boldsymbol{Rn} - \boldsymbol{nR} = -2I(\boldsymbol{n}\times\boldsymbol{R})\,. \qquad (13)$$

Also, as with quaternions

$$\boldsymbol{nR}_{\parallel} = \boldsymbol{R}_{\parallel}\boldsymbol{n}$$

$$\boldsymbol{nR}_{\perp} = -\boldsymbol{R}_{\perp}\boldsymbol{n}\,.$$

Using these relations along with Eq. (11) gives

$$\boldsymbol{nRn} = \boldsymbol{n}(\boldsymbol{R}_{\parallel} + \boldsymbol{R}_{\perp})\boldsymbol{n} = \boldsymbol{R}_{\parallel}\boldsymbol{nn} - \boldsymbol{R}_{\perp}\boldsymbol{nn} = \boldsymbol{R}_{\parallel} - \boldsymbol{R}_{\perp} = 2\boldsymbol{R}_{\parallel} - \boldsymbol{R}$$

$$(14)$$

Using Eqs. (13) and (14) in Eq. (12) gives

$$\boldsymbol{R}' = cos\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)\boldsymbol{R} + sin\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)(2\boldsymbol{R}_{\parallel} - \boldsymbol{R}) + 2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)$$

$$(\boldsymbol{n}\times\boldsymbol{R})\,.$$

Using half angle identities again gives the Rodrigues formula

$$\boldsymbol{R}' = \boldsymbol{R}cos(\theta) + 2\,sin\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)\boldsymbol{R}_{\parallel} + (\boldsymbol{n}\times\boldsymbol{R})sin(\theta)$$

$$\boldsymbol{R}' = \boldsymbol{R}cos(\theta) + \big(1 - cos(\theta)\big)\boldsymbol{R}_{\parallel} + sin(\theta)(\boldsymbol{n}\times\boldsymbol{R})\,.$$

Also, from the vector product Eq. (10) if u and ***v*** are unit vectors separated by angle $\frac{\theta}{2}$ then

$$\boldsymbol{uv} = \boldsymbol{u}\cdot\boldsymbol{v} + I(\boldsymbol{u}\times\boldsymbol{v}) = cos\left(\frac{\theta}{2}\right) + Isin\left(\frac{\theta}{2}\right)\boldsymbol{n}$$

$$\boldsymbol{vu} = \boldsymbol{u}\cdot\boldsymbol{v} - I(\boldsymbol{u}\times\boldsymbol{v}) = cos\left(\frac{\theta}{2}\right) - Isin\left(\frac{\theta}{2}\right)\boldsymbol{n}\,,$$

where again the axis of rotation ***n*** is the unit vector in the ($\boldsymbol{u}\times v$) direction.

Similar to the quaternion case, rotation of vector ***R*** about axis ***n*** by angle θ is then

$$\boldsymbol{R}' = \left(cos\left(\frac{\theta}{2}\right) - Isin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right)\boldsymbol{R}\left(cos\left(\frac{\theta}{2}\right) + Isin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right) = \boldsymbol{vuRuv}\,.$$

Conclusion

The use of complex numbers to represent vectors has computational advantages. Such a representation in 3-D has been historically accomplished using quaternions. However, the quaternion approach is not unique and has been seen as confusing. A Clifford algebra definition of complex numbers in representing 3-D vectors is equivalent to the traditional, and historically confusing, use of the quaternion definition. Given the fact that using the quaternion definition a vector is a purely imaginary quantity whose square is the negative of its squared length, while using the Clifford definition a vector is a real quantity whose square is positive, it may be preferable to use the Clifford definition for complex numbers in many applications [1].

Conflict of Interest Statement

Author has no conflict of interest regarding this article.

Reference

1. John Voight (2021) Quaternion Algebras. Springer https://link.springer.com/book/10.1007/978-3-030-56694-4.