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Almost Sure Limit Points for Continuous Time Random Walks

Gooty Divanji

Department of Statistics, Manasagangothri, University of Mysore, Mysore, India

ABSTRACT

Here we obtain almost sure limit points for a properly normalized partial sum continuous time random walk, where a continuous time random walk means a random walk subordinated to a renewal process. Continuous time random walks are used in physics to model anomalous diffusion.

Corresponding author

Gooty Divanji, Department of Statistics, Manasagangothri, University of Mysore, Mysore, India. Email: gootydivan@yahoo.co.in

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Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent positive-valued stable random variables (r.v.'s), with index α, $0 < \alpha < 1$, with a common characteristic function given by

$$E\left(e^{itX_1}\right) = \exp\left\{-|t|^{\alpha}\left(1 - i\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}$$

When the X_n's are independent identically distributed (i.i.d.) symmetric stable r.v.'s, Chover (1966) established the law of iterated logarithm (LIL) for partial sums, by normalizing the power. He showed that

$$\limsup_{n \to \infty} \left|\frac{S_n}{n^{\frac{1}{\alpha}}}\right|^{\frac{1}{\log\log n}} = e^{\frac{1}{\alpha}} \text{ almost sure (a.s.)}$$

Let $\{X_i, J_i, i \geq 1\}$ be a sequence of i.i.d. r.v's. Let $S_n = X_1 + X_2 + \ldots + X_n$ and $T_n = J_1 + J_2 + \ldots + J_n$. Let $N_t = \max\{n \geq 0: T_n \leq t\}$ be the renewal process of J_i.

A continuous time random walk (CTRW) is defined by

$Y_t = S_{N_t} = \sum_{i=1}^{N_t} X_i$. Here Xi represents a particle jump and Ji > 0 is the waiting time preceding that jump so that S_n represents the particle location after n jumps and T_n is the time of the n^{th} jump. Then N_t is the number of jumps by time $t > 0$, and the CTRW Y_t represents the particle location at time $t > 0$, which is a random walk subordinated to a renewal process, as used in physics to model anomalous diffusion. CTRW modules and the associated fractional diffusion equations are important in physics, hydrology, and finance applications In applications to finance, the particle jumps are price changes or long-term trades separated by a random waiting time between trades. For more information on applications of CTRW, see the references in Hwang and Wang (2012).

Hwang and Wang (2012) and Wang (2017) obtained Chover's form of LIL for a CTRW with jumps and waiting times in the domain of attraction of stable laws. That they established the limit supremum and limit infimum for a CTRW with jumps and waiting times in Chover's form. The purpose of this work is to obtain a.s. limit points for a CTRW with jumps in Chover's form for positive stable r.v.'s and to extend this to study the boundary-crossing problem.

In the next section, we present some known results. In section 3, we establish a.s. limit points for a CTRW in Chover's form, and in the last section, we extend this to the boundary-crossing problem.

Throughout the paper, ε, C, δ, and k, with or without a suffix or super suffix, stand for positive constants, with k and n confined to being positive integers. The abbreviations i.o., a.s., and d.f. stand for infinitely often, almost surely (or almost sure) and distribution function respectively, and $f(x) \sim g(x)$ stands for $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$

Som Known Results

Lemma 1: $\liminf_{t \to \infty} (\sup) \left(\frac{Y_t}{t^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\log\log n}} = 1 (e^{\frac{1}{\alpha}})$ a.s.

For proof, see corollary 1.2 on page 2 of Hwang and Wang (2012) and corollary 1.2 on page 960 of Wang (2017).

The following result of Hwang and Wang (2012), which is theorem 2.1 on page 3, plays a key role in establishing our result; hence, we state this result without proof.

Lemma 2: Let $\{J_i\}$ be a sequence of i.i.d. non-negative r.v.'s with a common d.f. G. Assume that G is absolutely continuous and that $1 - G(x) \sim C\, x^{-\beta}$, $0 < \beta < 1$, where C is some positive constant. Then we have

$$\liminf_{n\to\infty}\left(\frac{T_n}{n^{\frac{1}{\alpha}}}\right)^{\frac{1}{\log\log n}} = 1 \text{ a.s.}$$

Proof: To prove the result, it is enough to show that for all $\varepsilon > 0$,

$$P\left(T_n \le (\log n)^{-\varepsilon} n^{\frac{1}{\beta}} \text{ i.o.}\right) = 0 \qquad (1)$$

and

$$P\left(T_n \le (\log n)^{\varepsilon} n^{\frac{1}{\beta}} \text{ i.o.}\right) = 1 \qquad (2)$$

For the proof of (1) and (2), see Hwang and Wang (2012).

Main Result

Theorem 1

Let $\{X_i, i \ge 1\}$ be a sequence of i.i.d. positive valued stable r.v.'s with a common d.f. F that satisfies $1-F(x) \sim C\,x^{-\alpha}$, $0 < \alpha < 1$. Let $\{J_i, i \ge 1\}$, independent of $\{X_i, i \ge 1\}$, be a sequence of i.i.d. positive valued stable r.v.'s with a common d.f. G such that G is absolutely continuous and $1 - G(x) \sim C\, x^{-\beta}$, $0 < \beta < 1$, and for some $C > 0$. Then all

points in $\left[1, e^{\frac{1}{\alpha}}\right]$ are a.s. limit points of the sequence $\{\xi_t, t>0\}$,

where $\xi_t = \left(\frac{Y_t}{t^{\frac{\beta}{\alpha}}}\right)^{\frac{1}{\alpha}}$

Proof: Let an arbitrary point in $\left[1, e^{\frac{1}{\alpha}}\right]$ be $e^{\frac{u}{\alpha}}$, $0 \le u \le 1$ We can observe that the limit set of the sequence $\{\xi_t, t > 0\}$ is contained in $\left[1, e^{\frac{1}{\alpha}}\right]$, which is immediate from lemma 1. Hence the proof will be complete once we establish that every element in $(1, e^{1/\alpha})$ is a limit point of $\{\xi_t, t>0\}$. In other words for $e^{\frac{u}{\alpha}} \in \left(1, e^{\frac{1}{\alpha}}\right)$ with $0 < u < 1$ and for any ε_t, $0 < \varepsilon < 1$, it suffices to show that

$$P\left(Y_t \ge t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u+\varepsilon}{\alpha}} \text{ i.o.}\right) = 0 \qquad (3)$$

and

$$P\left(Y_t \ge t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u-\varepsilon}{\alpha}} \text{ i.o.}\right) = 1 \qquad (4)$$

First, we show (3). Let $t_k = \theta^k$, $0 < \theta < 1$. Here we introduce some notation for simplicity's sake and we follow Hwang and Wang's (2012) methodology to prove our result.

$$H_k = \left\{S_{N_{t_k}}\ t_k^{\frac{\beta}{\alpha}}\ (\log t_k)^{\frac{u+\varepsilon}{\alpha}}\right\},\ u_x = x^{\frac{1}{\beta}} (\log x)^{-\frac{\varepsilon}{2\beta}},\ v(x) = \sup\{y: u(y) \le x\},$$

$$L_k = \left\{S_{v(t_k)} \ge t_k^{\frac{\beta}{\alpha}}\ (\log t_k)^{\frac{u+\varepsilon}{\alpha}}\right\} \text{ and } R_k = \left\{N_{t_k} \ge v(t_k)\right\}.$$

Since $N_t = \max\{n \ge 0 : T_n \ge t\}$, and using lemma 2 equation (1), we have

$$P(R_k \text{ i.o.}) = P\left(T_{v(t_k)} \le t_k \text{ i.o.}\right) = P\left(T_{t_k} \le t_k^{\frac{\beta}{\alpha}}\ (\log t_k)^{\frac{u+\varepsilon}{\alpha}} \text{ i.o.}\right) = 0.$$

Now consider,

$$P(L_k) \le P\left(S_{v(t_k)} \ge \frac{t_k^{\frac{\beta}{\alpha}}}{(v(t_k))^{\frac{1}{\alpha}}}\ (\log t_k)^{\frac{u+\varepsilon}{\alpha}} (v(t_k))^{\frac{1}{\alpha}}\right)$$

Note that $u\left(t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{\varepsilon}{2}}\right) \sim t_k (\log t_k)^{\frac{\varepsilon}{2\beta}} \left(\log\left(t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{\varepsilon}{2}}\right)\right)^{-\frac{\varepsilon}{2\beta}} \ge u(v(t_k)) = t_k$.

Observing that u is increasing, we have, $t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{\varepsilon}{2\alpha}} \ge (v(t_k))^{\frac{1}{\alpha}}$

which implies

$$\frac{t_k^{\frac{\beta}{\alpha}}}{(v(t_k))^{\frac{1}{\alpha}}} \ge \frac{1}{(\log t_k)^{\frac{\varepsilon}{2\alpha}}}. \qquad (5)$$

Using (5) and the fact that X_i's are i.i.d. positive stable r.v.s, we get,

$$P(L_k) \le P\left(\frac{S_{v(t_k)}}{(v(t_k))^{\frac{1}{\alpha}}} \ge (\log t_k)^{\frac{u+\frac{\varepsilon}{2}}{\alpha}}\right) = P\left(X_1 \ge (\log t_k)^{\frac{u+\frac{\varepsilon}{2}}{\alpha}}\right) \sim \frac{C_1}{\left((\log t_k)^{\frac{u+\frac{\varepsilon}{2}}{\alpha}}\right)^{\alpha}} \le \frac{C_1}{(\log t_k)^{u+\frac{\varepsilon}{2}}},$$

where C_1 is some positive constant. Therefore $\sum_{t>0} P(L_k) < \infty$.

By the Borel–Cantelli lemma, we get $P(L_k \text{ i.o.}) = 0$.

Notice that

$$P\left(\bigcup_{k=n}^{\infty} H_k\right) = P\left(\bigcup_{k=n}^{\infty} H_k \cap \bigcap_{k=n}^{\infty} R_k^c\right) + P\left(\bigcup_{k=n}^{\infty} H_k \cap \left(\bigcap_{k=n}^{\infty} R_k^c\right)^c\right)$$
$$\le P\left(\bigcup_{k=n}^{\infty} L_k\right) + P\left(\bigcup_{k=n}^{\infty} R_k\right)$$

and by letting $n \to \infty$, we get $P(H_k \text{ i.o.}) \le P(L_k \text{ i.o.}) + P(R_k \text{ i.o.}) = 0$,

which implies

$$P\left(Y_{t_k} \ge t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{u+\varepsilon}{\alpha}} \text{ i.o.}\right) = 0 \text{ (or) } \limsup_{k\to\infty} t_k^{-\frac{\beta}{\alpha}} (\log t_k)^{-\frac{u+\varepsilon}{\alpha}} Y_{t_k} \le 1 \text{ a.s.} \qquad (6)$$

Since $t_k = \theta^k$, $0 < \theta < 1$, then

$$\limsup_{t\to\infty} t^{-\frac{\beta}{\alpha}} (\log t)^{-\frac{u+\varepsilon}{\alpha}} Y_t \le \limsup_{k\to\infty} \max_{t_{k-1} \le t \le t_k} t_k^{-\frac{\beta}{\alpha}} (\log t_k)^{-\frac{u+\varepsilon}{\alpha}} Y_{t_k}$$
$$\le \theta^{-\frac{\beta}{\alpha}} \limsup_{k\to\infty} t_k^{-\frac{\beta}{\alpha}} (\log t_k)^{-\frac{u+\varepsilon}{\alpha}} Y_{t_k} \text{ a.s.} = \theta^{-\frac{\beta}{\alpha}}, \text{ by (6).} \qquad (7)$$

This immediately yields (3) by letting $\theta \downarrow 1$.

Now we show (4). Let . $t_k = e^{k^{1+\delta}}$, $\delta > 0$. To show (4), we need to prove

$$\limsup_{k\to\infty} t_k^{-\frac{\beta}{\alpha}} (\log t_k)^{-\frac{u-\varepsilon}{\alpha}} Y_{t_k} \ge 1 \text{ a.s.} \qquad (8)$$

$$H_{1k} = \left\{S_{N_{t_k}} \ge t_k^{\frac{\beta}{\alpha}}\ (\log t_k)^{\frac{u-\varepsilon}{\alpha}}\right\},\ u_1(x) = x^{\frac{1}{\beta}} (\log x)^{-\frac{\varepsilon}{2\beta}},\ v_1(x) = \sup\{y: u_1(y) \le x\},$$

$$L_{1k} = \left\{S_{v_1(t_k)} - S_{v_1(t_{k-1})} \ge t_k^{\frac{\beta}{\alpha}}\ (\log t_k)^{\frac{u-\varepsilon}{\alpha}}\right\} \text{ and } R_k = \left\{N_{t_k} \ge v_1(t_k)\right\}.$$

Since $N_t = \max\{n \geq 0: T_n \leq t\}$ and using lemma 2 equation (2), we have

$$P(R_k \text{ i.o.}) = P\left(T_{v_1(t_k)} \leq t_k \text{ i.o.}\right) = P\left(T_{t_k} \leq t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{u-\varepsilon}{\alpha}} \text{ i.o.}\right) = 1.$$

Now consider, $P(L_{1k}) = P\left(S_{v_1(t_k)} - S_{v_1(t_{k-1})} \geq t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{u-\varepsilon}{\alpha}}\right)$

$$\geq P\left(S_{v_1(t_k)} - S_{v_1(t_{k-1})} \geq \frac{t_k^{\frac{\beta}{\alpha}}}{(v_1(t_k) - v_1(t_{k-1}))^{\frac{1}{\alpha}}} (\log t_k)^{\frac{u-\varepsilon}{\alpha}} (v_1(t_k) - v_1(t_{k-1}))^{\frac{1}{\alpha}}\right)$$

Consider $\frac{(v_1(t_k))^{\frac{1}{\alpha}}}{(v_1(t_k) - v_1(t_{k-1}))^{\frac{1}{\alpha}}} = \frac{(v_1(t_k))^{\frac{1}{\alpha}}}{(v_1(t_k))^{\frac{1}{\alpha}}\left(1 - \frac{v_1(t_{k-1})}{v_1(t_k)}\right)^{\frac{1}{\alpha}}}$ using the fact that

$$\frac{t_{k-1}}{t_k} \to 0, \text{ as } k \to \infty, \quad \frac{v_1(t_{k-1})}{v_1(t_k)} = \frac{t_{k-1}^{\beta} (\log t_{k-1})^{\frac{\varepsilon}{2}}}{t_k^{\beta} (\log t_k)^{\frac{\varepsilon}{2}}} \to 0, \text{ as } k \to \infty.$$

$$\frac{(v_1(t_k))^{\frac{1}{\alpha}}}{(v_1(t_k) - v_1(t_{k-1}))^{\frac{1}{\alpha}}} \to 1, \text{ as } k \to \infty \qquad (9)$$

Also, observe that $\frac{t_k^{\frac{\beta}{\alpha}}}{(v_1(t_k))^{\frac{1}{\alpha}}} = \frac{t_k^{\frac{\beta}{\alpha}}}{\left(t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{\varepsilon}{2}}\right)^{\frac{1}{\alpha}}} = (\log t_k)^{-\frac{\varepsilon}{2\alpha}}.$

(10)

Hence (9) and (10) yield,

$$P(L_{1k}) \geq P\left(S_{v_1(t_k)} - S_{v_1(t_{k-1})} \geq (\log t_k)^{\frac{u-\frac{\varepsilon}{2}}{\alpha}} (v_1(t_k))^{\frac{1}{\alpha}}\right)$$

$$\geq P\left(\frac{S_{v_1(t_k)} - S_{v_1(t_{k-1})}}{(v_1(t_k))^{\frac{1}{\alpha}}} \geq (\log t_k)^{\frac{u-\frac{\varepsilon}{2}}{\alpha}}\right)$$

$$\geq P\left(X_1 \geq (\log t_k)^{\frac{u-\frac{\varepsilon}{2}}{\alpha}}\right) \sim \frac{C_2}{(\log t_k)^{u-\frac{\varepsilon}{2}}},$$

where C_2 is some positive constant and the fact that X_i's are i.i.d. positive stable r.v.s. Therefore $\sum_{t>0} P(L_{1k}) = \infty.$

Since the events $\{L_{1k}\}$ are independent, by the Borel–Cantelli lemma, we get $P(L_{1k} \text{ i.o.}) = 1$.

Notice that, $P\left(\bigcup_{k=n}^{\infty} H_{1k}\right) \geq P\left(\bigcup_{k=m}^{\infty} H_{1k} \cap R_{1k}\right)$

$$\geq P\left(\bigcup_{k=m}^{\infty}\left\{S_{v_1(t_k)} \geq t_k^{\frac{\beta}{\alpha}} (\log t_k)^{\frac{u-\varepsilon}{\alpha}}\right\}\right) P\left(\bigcap_{k=m}^{\infty} R_k\right)$$

$$\geq P\left(\bigcup_{k=m}^{\infty} L_{1k}\right) P\left(\bigcap_{k=m}^{\infty} R_{1k}\right).$$

By letting $m \to \infty$, we get, $P(H_{1k} \text{ i.o.}) \geq P(L_{1k} \text{ i.o.}) P(R_{1k} \text{ i.o.}) = 1$,

which implies (8), and the proof of the theorem is now completed.

Boundary Crossing Problem

Here we study the boundary-crossing r.v.'s related to the above theorem. Now, for any $\varepsilon > 0$, let

$$M_t(\varepsilon) = \begin{cases} 1, & \text{if } Y_t \geq t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u+\varepsilon}{\alpha}} \\ 0, & \text{otherwise} \end{cases}$$

Let $N(\varepsilon)$ be the partial sum of $M_t(\varepsilon)$ and $N_\infty(\varepsilon) = \sum_{t>0} M_t(\varepsilon)$.

Observe that $N_\infty(\varepsilon)$ denotes the number of times $t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u+\varepsilon}{\alpha}}$ crosses the boundary ε. By the above theorem (see (3)), we have

$P\left(Y_t \geq t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u+\varepsilon}{\alpha}} \text{ i.o.}\right) = 0$. Hence

$P(N_\infty(\varepsilon) < \infty) = 1$ or $N_\infty(\varepsilon)$ is a proper r.v.

Here we show that all the moments in $0 < \lambda \leq 1$ are finite for this proper r.v. This proper r.v., $N_\infty(\varepsilon)$ has been studied by various authors, including Slivka (1969) and Slivka and Savero (1970).

Theorem 2

Let Y_t, $t > 0$ be the particle location at time $t > 0$ and as defined above. Then for $\varepsilon > 0$ and for any λ, $0 < \lambda \leq 1$,

$$E N_\infty^{\lambda} < \infty, \text{ if } \sum_{t>0} t^{\lambda-1} P\left(Y_t \geq t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u+\varepsilon}{\alpha}}\right) < \infty.$$

Proof: First, we show that, for $\lambda = 1$, $EN_\infty(\varepsilon) < \infty$, and then we claim that the lower moments existence follows that of the higher moments. Observe that

$$E N_\infty(\varepsilon) = \sum_{t>0} P\left(Y_t \geq t^{\frac{\beta}{\alpha}} (\log t)^{\frac{u+\varepsilon}{\alpha}}\right).$$

Following similar steps to the proof of (6), we can find some constant $C_3 > 0$ such that

$EN_\infty(\varepsilon) < C_3\, \theta^{\frac{\beta}{\alpha}} < \infty$, since $0 < \theta < 1$.

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